

David vs Goliath (You against the Markets),

A Dynamic Programming Approach to Separate the Impact and Timing of Trading Costs

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1 Abstract

To trade, or not to trade, that is the question!
Whether an optimizer can yield the answer,
Against the spikes and crashes of markets gone wild.
To quench one's thirst before liquidity runs dry,
Or wait till the tide of momentum turns mild.

A trader's conundrum is whether (and how much) to trade during a given interval or wait for the next interval when the price momentum is more favorable to his direction of trading. We develop a fundamentally

different stochastic dynamic programming model of trading costs based on the Bellman principle of optimality and show that trading costs are a zero sum game. Built on a strong theoretical foundation, this model can provide insights to market participants by splitting the overall move of the security price during the duration of an order into the Market Impact (price move caused by their actions) and Market Timing (price move caused by everyone else) components. Plugging different distributions of prices and volumes into this framework can help traders decide when to bear higher Market Impact by trading more in the hope of offsetting the cost of trading at a higher price later. We derive formulations of this model under different laws of motion of the security prices. We start with a benchmark scenario and extend this to include multiple sources of uncertainty, liquidity constraints due to volume curve shifts and relate trading costs to the spread.

2 Introduction

The recent blockbuster book, *David and Goliath: Underdogs, Misfits, and the Art of Battling Giants* (Gladwell 2013), talks about the advantages of disadvantages, which in the legendary battle refers to (among other things) the nimbleness that David possesses, due to his smaller size and lack of armour, that comes in handy while defeating the massive and seemingly unbeatable Goliath. Despite the inspiring tone of the story, the efforts of the most valiant financial market participant, can seem puny and turn out to be inadequate, as it gets undone when dealing with the gargantuan and mysterious temperament of uncertainty in the markets. We aim to provide mechanisms that can aid participants and make their life easier when confronting the markets; but given the nature of uncertainty in the social sciences (Kashyap 2014a), any weapon will prove to be insufficient compared to the sling shot that delivered the fatal blow to Goliath, until perhaps, one can discern the ability to read the minds of all the market participants.

A trader's conundrum is whether (and how much) to trade during a given interval or wait for the next interval when the price momentum is more favorable to his direction of trading; perhaps more ornately described as:

To trade, or not to trade, that is the question!

Whether an optimizer can yield the answer

Against the spikes and crashes of markets gone wild.

To quench one's thirst before liquidity runs dry

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We develop a fundamentally different stochastic dynamic programming model of trading costs based on the Bellman principle of optimality. Built on a strong theoretical foundation, this model can provide insights to market participants by splitting the overall move of the security price during the duration of an order into the Market Impact (price move caused by their actions) and Market Timing (price move caused by everyone else) components. Plugging different distributions of prices and volumes into this framework can help traders

decide when to bear higher Market Impact by trading more in the hope of offsetting the cost of trading at a higher price later. We derive formulations of this model under different laws of motion of the security prices. We start with a benchmark scenario and extend this to include multiple sources of uncertainty, liquidity constraints due to volume curve shifts and relate trading costs to the spread.

The unique aspect of our approach to trading costs is a method of splitting the overall move of the security price during the duration of an order into two components (Collins and Fabozzi 1991; Treynor 1994; Yegerman and Gillula 2014). One component gives the costs of trading that arise from the decision process that went into executing that particular order, as captured by the price moves caused by the executions that comprise that order. The other component gives the costs of trading that arise due to the decision process of all the other market participants during the time this particular order was being filled. This second component is inferred, since it is not possible to calculate it directly (at least with the present state of technology and publicly available data) and it is the difference between the overall trading costs and the first component, which is the trading cost of that order alone. The first and the second component arise due to competing forces, one from the actions of a particular participant and the other from the actions of everyone else that would be looking to fulfill similar objectives.

Naturally, it follows that each particular participant can only influence to a greater degree the cost that arises from his actions as compared to the actions of others, over which he has lesser influence, but an understanding of the second component, can help him plan and alter his actions, to counter any adversity that might arise from the latter. Any good trader would do this intuitively as an optimization process, that would minimize costs over two variables direct impact and timing, the output of which recommends either slowing down or speeding up his executions. With this measure, traders now actually have a quantitative indicator to fine tune their decision process. When we decompose the costs, it would be helpful to try and understand how the two sub costs could vary as a proportion of the total. The volatility in these two components, which would arise from different sources (market conditions), would require different responses and hence would affect the optimization problem mentioned above invoking different sorts of handling and based on the situation, traders would know which cost would be the more unpredictable one and hence focus their efforts on minimizing the costs arising from that component.

The key innovation can be explained as follows: A jump up in price on an execution that comprises a buy order is considered adverse and attributed as impact, while a fall in price is not. Yes, the price could fall further if not for the backstop provided by the executions that comprise the buy order; but the key aspect to remember here is the bilateral nature of trading. A price fall for the buyer (or a benefit for him) is impact for the seller (and hence adverse); and the seller bears the cost in this case. Most trading cost models consider elaborate theories of the price drifting around, but what actually happens during the transfer of securities is one party, usually, has an upper hand and that is the portion we look to measure as impact for

the other party. The key fallout from measuring impact this way is that we have a better way to measure the effect of our actions from when we have a concrete advantage to when we are okay to put up with a certain disadvantage. While no measure of trading costs is perfect and complete, this methodology goes a long way in actually providing tangible ways for someone to understand the effect of their decision process and the associated implementation of trades.

Another analogy to understand this methodology is to think of each execution as effecting a state transition from one price level to another. The impact is then the cost or charge involved to make the state transition. We can also think of the change in price levels as moving from station to another in a train and the ticket price is the cost involved to make this journey. If there is excess demand to travel from one station to another, the ticket price which is the same for everyone, changes accordingly and only those that are willing to pay can make the journey. That we are considering the state transitions for each execution at millisecond intervals, means that we are building from the bottom up and aggregating smaller effects into an overall impact number for the order, based on the executions that comprise it. Theoretically since it is possible that multiple parties could execute simultaneously (two or more buyers and / or sellers on each side), the question of which of the parties is more responsible for causing the price level to change and whether there needs to be a proportional allocation of the price jump does not set in, since all the parties are travelers on the same journey and they all have to pay the ticket price. Though, for executions that happen through a continuous auction process at larger intervals of time, a proportional allocation based on the size of each parties execution might be a possible alternative and will be pursued in later papers.

Figures 1 and 2 show the reversion in the price after an order has completed, broken down by volatility and momentum buckets. The full order sample includes 148,812 orders from 70+ countries with 17 countries having atleast thousand orders each. The reversion is based on two measures:

1. In time, 5 minutes and 60 minutes after an order has completed.
2. In multiples of the order size, one times and five times the size of the order.

The five Trade Momentum buckets are based on the side adjusted percentage return during the order's trading interval:

1. Significant Adverse ($<-2\%$)
2. Adverse ($-1/3\%$ thru -2%)
3. Neutral ($-1/3\%$ thru $+1/3\%$)

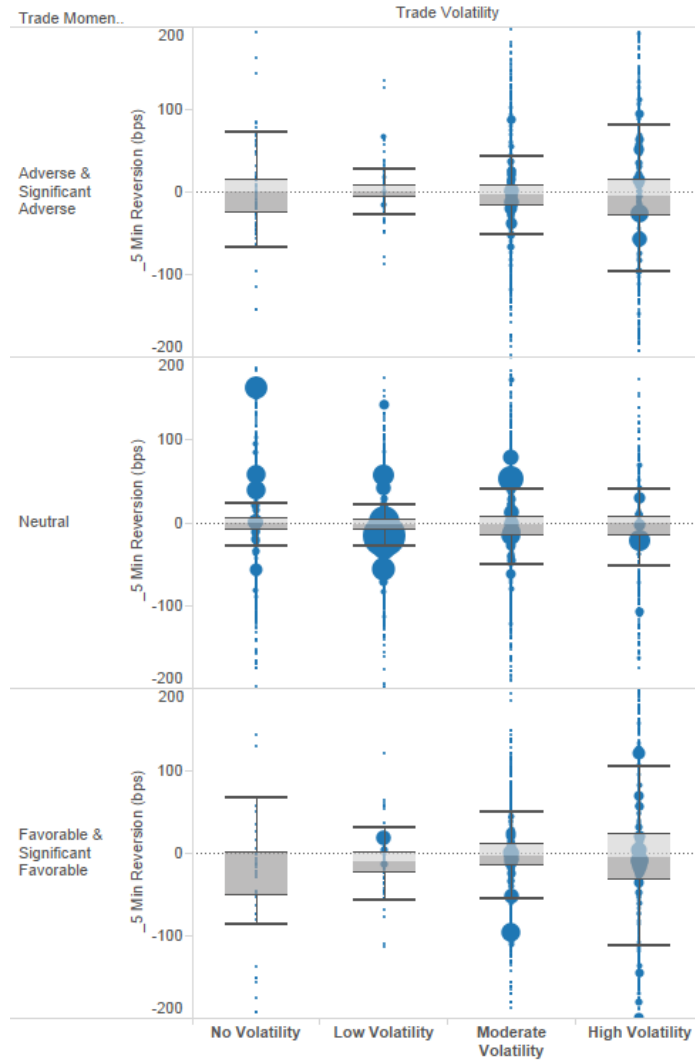
4. Favorable (+1/3% thru 2%)
5. Significant Favorable (>+2%)

The four Trade Volatility buckets are based on the coefficient of variation of prices during the execution horizon:

1. High Volatility (>0.0050)
2. Moderate Volatility (0.0010 thru 0.0050)
3. Low Volatility (0.000000000000001 thru 0.0010)
4. No Volatility (≤ 0.000000000000001)

The size of the bubbles indicates the relative magnitude of the order and its position on the vertical axis signifies the reversion amount in basis points. The box and the whisker capture the areas where 25% and 75% of the sample resides. Not surprisingly, the momentum is higher in periods of greater volatility, as seen more clearly from the measures based on multiples of the order size. The higher volatility accentuates the efforts required to trade in such an environment. This illustrates the issue that traders face and the optimization process that is followed, where they try to benefit from positive momentum and try to avoid adverse momentum by trading more when adverse momentum is anticipated, while being conscious of the level of volatility.

Reversion Order Distribution 5 Min



Reversion Order Distribution Multiple 1

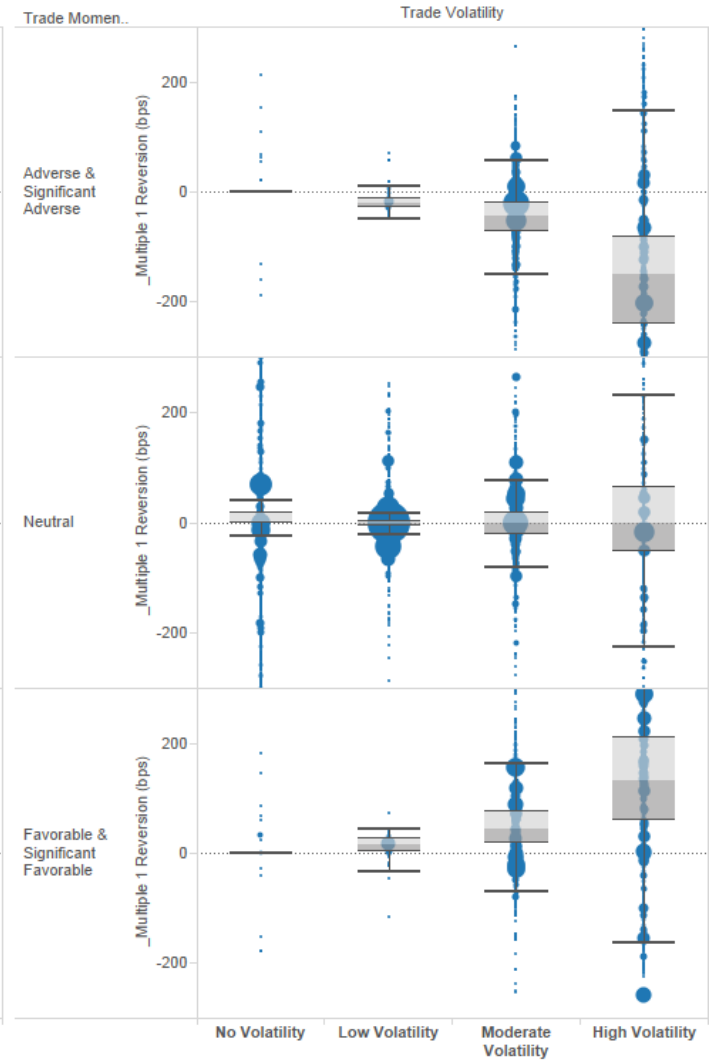
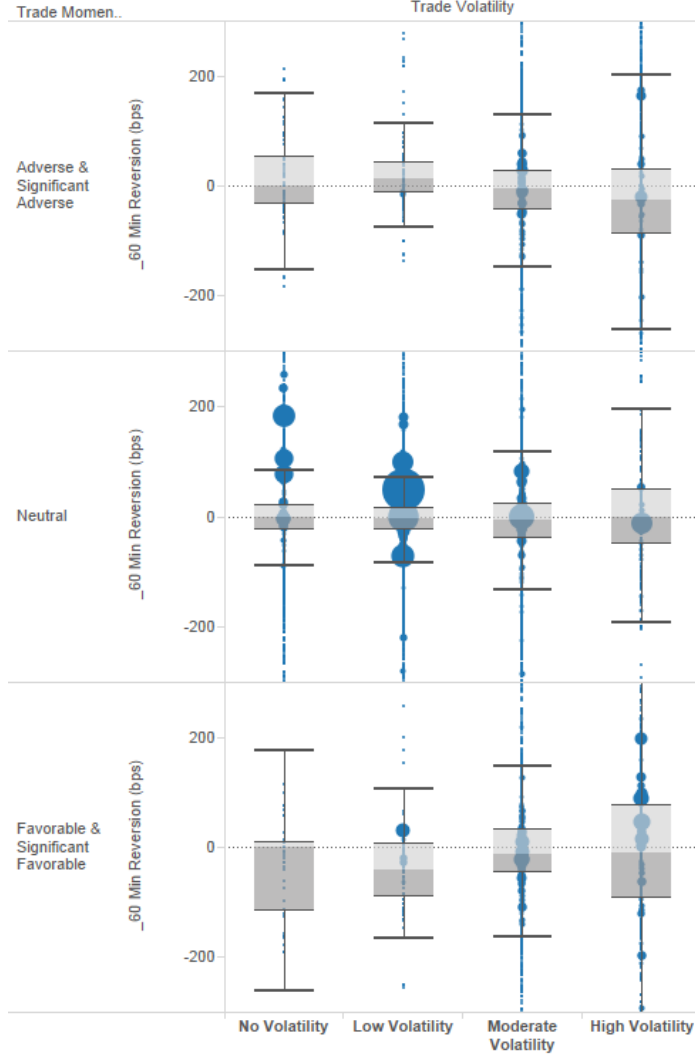


Figure 1: Reversion Distributions by Momentum and Volatility Environments - Shorter Horizon

Reversion Order Distribution 60 Min



Reversion Order Distribution Multiple 5

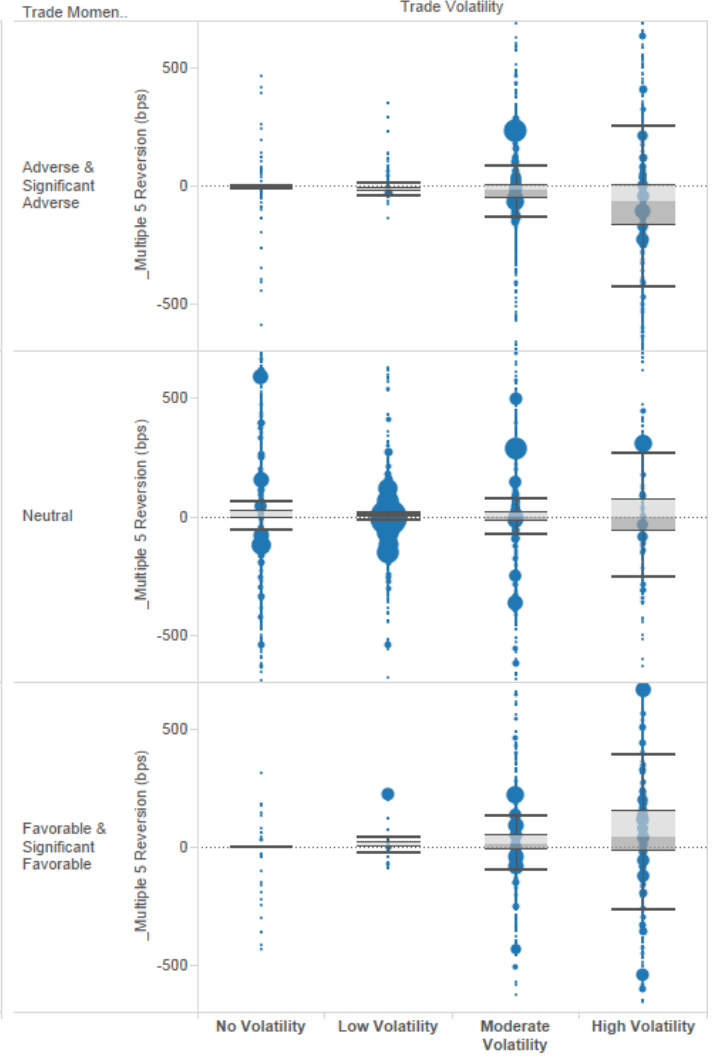


Figure 2: Reversion Distributions by Momentum and Volatility Environments - Longer Horizon

2.1 Related Literature

Building on the foundation laid by (Bertsimas and Lo 1998), another popular way to decompose trading costs is into temporary and permanent impact [See Almgren and Chriss (2001); Almgren (2003); and Almgren, Thum, Hauptmann and Li (2005)]. While the theory behind this approach is extremely elegant and considers both linear and nonlinear functions of the variables for estimating the impact, a practical way to compute it requires measuring the price a certain interval after the order. This interval is ambiguous and could lead to lower accuracy while using this measure. More recent extensions include: (Huberman and Stanzl 2005) minimize the mean and variance of the costs of trading, for the case of market orders only, and derive explicit formulas for the optimal trading strategies, showing that risk-averse liquidity traders reduce their order sizes

over time and execute a higher fraction of their total trading volume in early periods when price volatility or liquidity increases. (Almgren and Lorenz 2007) derive optimal strategies where the execution accelerates when the price moves in the trader’s favor, and slows when the price moves adversely; (Kissell and Malamut 2006) term such adaptive strategies “aggressive-in-the-money”; A “passive-in-the-money” strategy would react oppositely.

(Schied and Schölkneborn 2009) use a stochastic control approach building upon the continuous time model of Almgren (2003) and show that the value function and optimal control satisfy certain nonlinear parabolic partial differential equations that can be solved numerically. (Schied, Schölkneborn and Tehranchi 2010) consider the problem faced by an investor who must liquidate a given basket of assets over a finite time horizon. They assume that the investor’s utility has constant absolute risk aversion (CARA) and that the asset prices are given by a very general continuous-time, multi-asset price impact model and show that the investor does no worse if he narrows his search to deterministic strategies. (Gatheral and Schied 2011) find a closed-form solution for the optimal trade execution strategy in the Almgren–Chriss framework assuming the underlying unaffected stock price (stock price before the impact or before the transaction occurs) process is a geometric Brownian motion; (Schied 2013) investigates the robustness of this strategy with respect to misspecification of the law of the underlying unaffected stock price process. (Brunnermeier and Pedersen 2005); and (Carlin, Lobo and Viswanathan 2007) are extensions to situations with several competing traders.

(Huberman and Stanzl 2004) provide theoretical arguments showing that in the absence of quasi-arbitrage (availability of a sequence of round-trip trades that generate infinite expected profits with an infinite Sharpe ratio - infinite expected profits per unit of risk), permanent price-impact functions must be linear; though empirical investigations suggest that the shape of the limit order book (LOB) can be more complex (Hopman 2007).

In contrast to many studies, where the dynamics of the asset price process is taken as a given fundamental, (Obizhaeva and Wang 2013) proposed a market impact model that derives its dynamics from an underlying model of a LOB. In this model, the ask part of the LOB consists of a uniform distribution of shares offered at prices higher than the current best ask price. When the large trader is not active, the mid price of the LOB fluctuates according to the actions of noise traders, and the bid-ask spread remains constant. A buy market order of the large trader, however, consumes a block of shares located immediately to the right of the best ask and thus increase the ask price by a linear proportion of the size of the order. In addition, the LOB will recover from the impact of the buy order, i.e., it will show a certain resilience. The resulting price impact will neither be instantaneous nor entirely permanent but will decay on an exponential scale. (Alfonsi, Fruth and Schied 2010) extend this by allowing for a general shape of the LOB defined via a given density function, which can accommodate empirically observed LOB shapes and obtain a nonlinear price impact of market orders. Empirical studies based on the LOB model are (Biais, Hillion and Spatt 1995; Potters and

Bouchaud 2003; Bouchaud, Gefen, Potters and Wyart 2004; and Weber and Rosenow 2005).

A related strand of literature looks at models of the LOB from the perspective of dealers seeking to submit optimal strategies (maximize the utility of total terminal wealth) of bid and ask orders. Ho and Stoll (1981) analyse the optimal prices for a monopolistic dealer in a single stock when faced with a stochastic demand to trade, modelled by a continuous time Poisson jump process, and facing return uncertainty, modelled by diffusion processes. Ho and Stoll (1980), consider the problem of dealers under competition (each dealer's pricing strategy depends not only on his own current and expected inventory position and his other characteristics but also on the current and expected inventory and other characteristics of the competitor) and show that the bid and ask prices are shown to be related to the reservation (or indifference) prices of the agents.

Avellaneda and Stoikov (2008) combine the utility framework with the microstructure of actual limit order books as described in the econophysics literature to infer reasonable arrival rates of buy and sell orders; Du, Zhu and Zhao (2016) extend the price dynamics to follow a Geometric Brownian Motion (GBM) in which the drift part is updated by Bayesian learning in the beginning of the transaction day to capture the trader's estimate of other traders' target sizes and directions. Cont, Stoikov and Talreja (2010) describe a stylized model for the dynamics of a limit order book, where the order flow is described by independent Poisson processes and estimate the model parameters from high-frequency order book time-series data from the Tokyo Stock Exchange. While our work focuses on separating impact and timing in the (Bertsimas and Lo 1998) framework; a natural and interesting continuation would be to extend this separation to models of the limit order book.

Models of market impact and the design of better trading strategies are becoming an integral part of the present trend at automation and the increasing use of algorithms. (Jain 2005) assembles the dates of announcement and actual introduction of electronic trading by the leading exchange of 120 countries to examine the long term and medium term impact of automation. He finds that automation of trading on a stock exchange has a long-term impact on listed firms' cost of equity. Estimates from dividend growth model as well as international CAPM suggest a significant decline in expected returns after the introduction of electronic trading in the world's equity markets, especially in the developing nations and confirms the finding from previous studies that electronic trading improves a stock's liquidity and reduces investors' trading costs. (Hendershott, Jones and Menkveld 2011) perform an empirical study on New York Stock Exchange stocks and find that algorithmic trading and liquidity are positively related.

It is worth noting a contrasting result from an earlier study. (Venkataraman 2001) compares securities on the New York Stock Exchange (NYSE) (a floor-based trading structure with human intermediaries, specialists, and floor brokers) and the Paris Bourse (automated limit-order trading structure). He finds that execution costs might be higher on automated venues even after controlling for differences in adverse selection, relative

tick size, and economic attributes. A trade occurs when an aggressive trader submits a market order and demands liquidity, hence the rules on a venue are designed to attract demanders of liquidity and nudge liquidity providers to display their orders. Displaying limit orders involves risks. First, the counter-parties could be better informed, and liquidity providers could get picked off. Hence, they would like the trading system to allow them to trade selectively with counter-parties of their choice. Second, they risk being front-run by other traders with an increase in the market impact of their orders. Hence, large traders want to hide their orders and expose them only to traders who are most likely to trade with them. This means fully automated exchanges, which anecdotally seems to be the way ahead, need to take special care to formulate rules to help liquidity providers better control the risks of order exposure. What this also means is that the design of better strategies and models is crucial to survive and thrive in this continuing trend at automation.

3 Methodological Fundamentals

3.1 Terminology and Pseudo Mathematical Expressions

We now introduce some terminology used throughout the discussion.

1. Total Slippage - The overall price move on the security during the order duration. This is also a proxy for the implementation shortfall (Perold 1988 and Treynor 1981). It is worth mentioning that there are many similar metrics used by various practitioners and this concept gets used in situations for which it is not the best suited (Yegerman and Gillula 2014). While the usefulness of the Implementation Shortfall, or slippage, as a measure to understand the price shortfalls that can arise between constructing a portfolio and while implementing it, is not to be debated, slippage need to be supplemented with more granular metrics when used in situations where the effectiveness of algorithms or the availability of liquidity need to be gauged.
2. Market Impact (MI) - The price moves caused by the executions that comprise the order under consideration. In short, the MI is a proxy for the impact on the price from the liquidity demands of an order. This metric is generally negative (by using a convention to show it as a cost; below we consider it as positive quantity for simplicity) or zero since in most cases, the best impact we can have is usually no impact.
3. Market Timing - The price moves that happen due to the combined effect of all the other market participants during the order duration.
4. Market Impact Estimate (MIE) - This is the estimate of the Market Impact, explained in point two above, based on recent market conditions. The MIE calculation is the result of a simulation which considers the number of executions required to fill an order and the price moves encountered while

filling this order, depending on the market micro-structure as captured by the trading volume and the price probability distribution including upticks and down-ticks, over the past few days. This simulation can be controlled with certain parameters that dictate the liquidity demanded on the order, the style of trading, order duration, market conditions as reflected by start of trading and end of trading times. In short, the MIE is an estimated proxy for the impact on the price from the liquidity demands of an order. Such an approach holds the philosophical viewpoint that making smaller predictions and considering their combined effect would result in lesser variance as opposed to making a large prediction; estimations done over a day as compared to estimations over a month, say. A geometrical intuition would be that fitting more lines (or curves) over a set of points would reduce the overall error as compared to fitting lesser number of lines (or curves) over the same set of points. When combining the results of predictions, of course, we have to be mindful of the errors of errors, which can get compounded and lead the results astray, and hence, empirical tests need to be done to verify the suitability of such a technique for the particular situation.

5. All these variables are measured in basis points to facilitate ease of comparison and aggregation across different groups. It is possible to measure these in cents per share and also in dollar value or other currency terms.
6. The following equations, expressed in pseudo mathematical terms to facilitate easier understanding, govern the relationships between the variables mentioned above. All the formal mathematics, including the stochastic dynamic programming model of trading costs, is relegated to later sections.

Total Slippage = Market Impact + Market Timing

{Total Price Slippage = Your Price Impact + Price Impact From Everyone Else (Price Drift)}

Market Impact Estimate = Market Impact Prediction = f (Execution Size, Liquidity Demand)

Execution Size = g (Execution Parameters, Market Conditions)

Liquidity Demand = h (Execution Parameters, Market Conditions)

Execution Parameters \leftrightarrow vector comprising (Order Size, Security, Side, Trading Style, Timing Decisions)

Market Conditions \leftrightarrow vector comprising (Price Movement, Volume Changes, Information Set)

Here, f , g , h are functions. We could impose concavity conditions on these functions, but arguably, similar results are obtained by assuming no such restrictions and fitting linear or non-linear regression coefficients, which could be non-concave or even discontinuous allowing for jumps in prices and volumes. The specific functional forms used could vary across different groups of securities or even across individual securities or even across different time periods for the same security. The crucial aspect of any such estimation is the comparison with the costs on real orders, as outlined earlier. Simpler modes are generally more helpful in

interpreting the results and for updating the model parameters. Hamilton [1994] and Gujarati [1995] are classic texts on econometric methods and time series analysis that accentuate the need for parsimonious models.

The Auxiliary Information Set could be anything under our Sun or even from under other heavenly objects. A useful variable to include would be the blood pressure and heart rate time series of a representative group of security traders.

4 Dynamic Recursive Trading Cost Model

A dynamic programming approach lends itself naturally to modeling optimal execution strategies. The initial excitement, of having stumbled upon a (seemingly) novel idea to apply dynamic programming to trading costs and the subsequent disappointment upon realizing that (Bertsimas and Lo 1998) had long ago embarked on this journey, was later replaced by awe and appreciation at the elegance and completeness of their treatment. They start with a simple arithmetic random walk for the law of motion of prices and later extend it to a geometric Brownian motion. Closed form solutions for many scenarios and numerical solutions follow quite easily.

Existing dynamic programming methods to optimizing trading costs and execution scheduling are of limited use to practitioners and traders since they do not provide a way for them to understand how their actions at each stage would affect the price (as opposed to the combined effect of everyone else or the market) and thereby pointing out specific aspects of the system that they can hope to influence (Kashyap 2014b, 2015a). Hence, we start with the benchmark dynamic programming problem and modify the reward function in the Bellman equation to suit our innovation.

4.1 Notation for Optimal Trading using a Dynamic Programming Approach

- \bar{S} , the total number of shares that need to be traded.
- T , the total duration of trading.
- N , the number of trading intervals.
- $\tau = T/N$, the length of each trading interval. We assume the time intervals are of the same duration, but this can be relaxed quite easily. In continuous time, this becomes, $N \rightarrow \infty, \tau \rightarrow 0$.
- The time then becomes divided into discrete intervals, $t_k = k\tau$, $k = 0, \dots, N$.
- For simplicity, let time be measured in unit intervals giving, $t = 1, 2, \dots, T$.

- S_t , the number of shares acquired in period t at price P_t .
- P_0 can be any reference price or benchmark used to measure the slippage. It is generally taken to be the arrival price or the price at which the portfolio manager would like to complete the purchase of the portfolio.
- Our objective is to formulate a trading trajectory, or a list of total pending shares, W_1, \dots, W_{T+1} . Here, W_t is the number of units that we still need to trade at time t . This would mean, $W_1 = \bar{S}$ and $W_{T+1} = 0$ implies that \bar{S} must be executed by period T . Clearly, $\bar{S} = \sum_{j=1}^T S_j$. This can equivalently be represented by the list of executions completed, S_1, \dots, S_T . Here, $W_t = W_{t-1} - S_{t-1}$ or $S_{t-1} = W_{t-1} - W_t$ is the number of units traded between times $t-1$ and t . W_t and S_t are related as below.

$$W_t = \bar{S} - \sum_{j=1}^{t-1} S_j = \sum_{j=t}^T S_j, \quad t = 1, \dots, T.$$

4.2 Benchmark Dynamic Programming Model

This is the simplest scenario where the trader would try to minimize the overall acquisition value of his holdings. In this case, securities are being bought. It is then logical to set a no sales constraint when the objective is to buy securities. The rest of treatment applies with simple modifications when securities are sold. The baseline objective function and constraints are written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^T S_t P_t \right]$$

$$\sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1}$$

The law of motion of price, P_t can be written as

$$P_t = P_{t-1} + \theta S_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0$$

$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv$ Zero Mean IID (Independent Identically Distributed) random shock or white noise

The law of motion includes two distinct components: the dynamics of P_t in the absence of our trade, (the trades of other may be causing prices to fluctuate) and the impact that our trade of S_t shares has on the execution price P_t . This simple price change relationship assumes that the former component is given by an arithmetic random walk and the latter component is a linear function of trade size so that a purchase of S_t shares may be executed at the prevailing price plus an impact premium of θS_t . Here, θ captures the effect of

transaction size on the price. In the absence of this transaction, the price process evolves as a pure arithmetic random walk. This then implies that from any participants view, the sum of all the price movements or the new price levels established by all other participants evolves as a random walk. For simplicity, we ignore the no sales constraint, $S_t \geq 0$.

The Bellman equation is based on the observation that a solution or optimal control $\{S_1^*, S_2^*, \dots, S_T^*\}$ must also be optimal for the remaining program at every intermediate time t . That is, for every t , $1 < t < T$ the sequence $\{S_t^*, S_{t+1}^*, \dots, S_T^*\}$ must still be optimal for the remaining program $E_t \left[\sum_{k=t}^T P_k S_k \right]$. The below relates the optimal value of the objective function in period t to its optimal value in period $t + 1$:

$$V_t(P_{t-1}, W_t) = \min_{\{S_t\}} E_t [P_t S_t + V_{t+1}(P_t, W_{t+1})]$$

4.3 Introducing our Innovation into the Implementation Shortfall

As a refresher, the total slippage or implementation shortfall is derived below with the understanding that we need to use the Expectation operator when we are working with estimates or future prices. (Kissell 2006) provides more details including the formula where the portfolio may be partly executed.

$$\text{Paper Return} = \bar{S}P_T - \bar{S}P_0$$

$$\text{Real Portfolio Return} = \bar{S}P_T - \left(\sum_{t=1}^T S_t P_t \right)$$

$$\begin{aligned} \text{Implementation Shortfall} &= \text{Paper Return} - \text{Real Portfolio Return} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S}P_0 \end{aligned}$$

This can be written as,

$$\begin{aligned} \text{Implementation Shortfall} &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S}P_0 \\ &= \left(\sum_{t=1}^T S_t P_t \right) - P_0 \left(\sum_{t=1}^T S_t \right) \\ &= S_1 P_1 + S_2 P_2 + \dots + S_T P_T - S_1 P_0 - S_2 P_0 - \dots - S_T P_0 \\ &= S_1 (P_1 - P_0) + S_2 (P_2 - P_0) + \dots + S_T (P_T - P_0) \end{aligned}$$

$$\begin{aligned}
\text{Implementation Shortfall} = & S_1 (P_1 - P_0) + \\
& S_2 (P_2 - P_1) + S_2 (P_1 - P_0) + \\
& S_3 (P_3 - P_2) + S_3 (P_2 - P_1) + S_3 (P_1 - P_0) + \\
& \dots + \\
& S_T (P_T - P_{T-1}) + S_T (P_{T-1} - P_{T-2}) + \dots + S_T (P_1 - P_0)
\end{aligned}$$

The innovation we introduce would incorporate our earlier discussion about breaking the total impact or slippage, Implementation Shortfall, into the part from the participants own decision process, Market Impact, and the part from the decision process of all other participants, Market Timing. This Market Impact, would capture the actions of the participant, since at each stage the penalty a participant incurs should only be the price jump caused by their own trades and that is what any participant can hope to minimize. A subtle point is that the Market Impact portion need only be added up when new price levels are established. If the price moves down and moves back up (after having gone up once earlier and having been already counted in the Impact), we need not consider the later moves in the Market Impact (and hence implicitly left out from the Market Timing as well). This alternate measure would only account for the net move in the prices but would not show the full extent of aggressiveness and the push and pull between market participants and hence is not considered here, though it can be useful to know and can be easily incorporated while running simulations. We discuss two formulations of our measure of the Market Impact in the next two subsections. The reason for calling them simple and complex will become apparent as we continue the discussion.

4.3.1 Market Impact Simple Formulation

The simple market impact formulation does not consider the impact of the new price level established on all the future trades that are yet to be done. From a theoretical perspective it is useful to study this since it provides a closed form solution and illustrates the immense practical application of separating impact and timing. This approach can be a useful aid in markets that are clearly not trending and where the order size is relatively small compared to the overall volume traded, ensuring that any new price level established does not linger on for too long and prices gets reestablished due to the trades of other participants. This property is akin to checking that shocks to the system do not take long to dissipate and equilibrium levels (or rather new pseudo equilibrium levels) are restored quickly. Our measure of the Market Impact then becomes,

$$\text{Market Impact} = \sum_{t=1}^T \{\max [(P_t - P_{t-1}), 0] S_t\}$$

The Market Timing is then given by,

$$\begin{aligned}\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - \sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] S_t \}\end{aligned}$$

For illustration, let us consider some examples,

1. When all the successive price moves are above their corresponding previous price, that is $\max[(P_t - P_{t-1}), 0] = (P_t - P_{t-1})$, we have

$$\begin{aligned}\text{Market Impact} &= \sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] S_t \} \\ &= S_1 (P_1 - P_0) + S_2 (P_2 - P_1) + S_3 (P_3 - P_2) + \dots + S_T (P_T - P_{T-1})\end{aligned}$$

$$\begin{aligned}\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - S_1 (P_1 - P_0) - S_2 (P_2 - P_1) - S_3 (P_3 - P_2) - \dots - S_T (P_T - P_{T-1}) \\ &= S_1 P_0 + S_2 P_1 + S_3 P_2 + \dots + S_T P_{T-1} - \bar{S} P_0 \\ &= S_2 (P_1 - P_0) + S_3 (P_2 - P_0) + \dots + S_T (P_{T-1} - P_0)\end{aligned}$$

2. Some of the successive prices are below their corresponding previous price, let us say, $(P_2 < P_1)$ and $(P_3 < P_2)$, we have

$$\begin{aligned}\text{Market Impact} &= \sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] S_t \} \\ &= S_1 (P_1 - P_0) + S_2 (0) + S_3 (0) + \dots + S_T (P_T - P_{T-1})\end{aligned}$$

$$\begin{aligned}\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - S_1 (P_1 - P_0) - S_2 (0) - S_3 (0) - \dots - S_T (P_T - P_{T-1}) \\ &= S_2 P_2 + S_3 P_3 + S_1 P_0 + S_4 P_3 + S_5 P_4 + \dots + S_T P_{T-1} - \bar{S} P_0 \\ &= S_2 (P_2 - P_0) + S_3 (P_3 - P_0) + S_4 (P_3 - P_0) + S_5 (P_4 - P_0) + \dots + S_T (P_{T-1} - P_0)\end{aligned}$$

4.3.2 Market Impact Complex Formulation

Another measure of the Market Impact can be formulated as below which represents the idea that when a participant seeks liquidity and establishes a new price level, all the pending shares or the unexecuted program is affected by this new price level. This is a more realistic approach since the action now will explicitly affect the shares that are not yet executed. This measure can be written as,

$$\text{Market Impact} = \sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] W_t\}$$

The Market Timing is then given by,

$$\begin{aligned} \text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - \sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] W_t\} \end{aligned}$$

For illustration, let us consider some examples,

1. When all the successive price moves are above their corresponding previous price, that is $\max[(P_t - P_{t-1}), 0] = (P_t - P_{t-1})$, we have

$$\begin{aligned} \text{Market Impact} &= \sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] W_t\} \\ &= W_1 (P_1 - P_0) + W_2 (P_2 - P_1) + W_3 (P_3 - P_2) + \dots + W_T (P_T - P_{T-1}) \end{aligned}$$

$$\begin{aligned} \text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\ &= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - W_1 (P_1 - P_0) - W_2 (P_2 - P_1) - W_3 (P_3 - P_2) - \dots - W_T (P_T - P_{T-1}) \\ &= \left[\sum_{t=1}^T (W_t - W_{t+1}) P_t \right] - W_1 P_0 - W_1 (P_1 - P_0) \\ &\quad - W_2 (P_2 - P_1) - W_3 (P_3 - P_2) - \dots - W_T (P_T - P_{T-1}) \\ &= (W_1 - W_2) P_1 + (W_2 - W_3) P_2 + \dots + (W_T - W_{T+1}) P_T \\ &\quad - W_1 P_0 - W_1 (P_1 - P_0) - W_2 (P_2 - P_1) - W_3 (P_3 - P_2) - \dots - W_T (P_T - P_{T-1}) \\ &= 0 \end{aligned}$$

2. Some of the successive prices are below their corresponding previous price, let us say, $(P_2 < P_1)$ and $(P_3 < P_2)$,

we have

$$\begin{aligned}
\text{Market Impact} &= \sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] W_t\} \\
&= W_1 (P_1 - P_0) + W_2 (0) + W_3 (0) + \dots + W_T (P_T - P_{T-1})
\end{aligned}$$

$$\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= \left(\sum_{t=1}^T S_t P_t \right) - \bar{S} P_0 - W_1 (P_1 - P_0) - W_2 (0) - W_3 (0) - \dots - W_T (P_T - P_{T-1}) \\
&= \left[\sum_{t=1}^T (W_t - W_{t+1}) P_t \right] - W_1 P_0 - W_1 (P_1 - P_0) \\
&\quad - W_2 (0) - W_3 (0) - \dots - W_T (P_T - P_{T-1}) \\
&= (W_1 - W_2) P_1 + (W_2 - W_3) P_2 + \dots + (W_T - W_{T+1}) P_T \\
&\quad - W_1 P_0 - W_1 (P_1 - P_0) - W_2 (0) - W_3 (0) - \dots - W_T (P_T - P_{T-1}) \\
&= -W_2 P_1 + W_2 P_2 - W_3 P_2 + W_3 P_3 \\
&= W_2 (P_2 - P_1) + W_3 (P_3 - P_2)
\end{aligned}$$

4.3.3 Trading Costs as a Zero Sum Game

It should be easily evident that in a given interval, the sum of market impact and the sum of market timing across all market participants equals zero. This is immediately obvious in the case that there are only two participants and there is only one single interval, since negative implementation shortfall for the buyer shows up as positive implementation shortfall for the seller; the impact for the buyer shows up as timing for the seller and vice versa. We note that the total amount bought in any interval is equal to the total amount sold. When there are more than two participants and multiple intervals, if we consider the actions in each interval and add up the impact and timing figures across everyone, it shows the zero sum nature of the trading game (For different types of zero sum games and methods of solving them, see Brown 1951; Gale, Kuhn and Tucker 1951; Von Neumann and Morgenstern 1953; Von Neumann 1954; Rapoport 1973; Crawford 1974; Laraki and Solan 2005; Hamadi, 2006); (Bodie and Taggart 1978; Bell and Cover 1980; Turnbull 1987; Hill 2006; Chirinko and Wilson 2008 consider zero sum games in the financial context). The result holds for both the simple and complex formulations of market impact.

Proposition 1. *Trading costs are a zero sum game. The sum of market impact and market timing across*

all participants, in any given time interval, should equal zero.

$$\text{Total Market Impact} + \text{Total Market Timing} = 0$$

Proof. See Appendix 7.1. □

4.4 Alternative Dynamic Market Impact Model

4.4.1 Simple Formulation

Incorporating the Simple Market Impact formulation from the previous section, the objective function and the Bellman equation can be modified as,

$$\begin{aligned} \min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] S_t \} \right] \\ \sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1} \\ P_t = P_{t-1} + \theta S_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0, \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \end{aligned}$$

The Bellman equation then becomes,

$$V_t(P_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max\{(P_t - P_{t-1}), 0\} S_t + V_{t+1}(P_t, W_{t+1})]$$

By starting at the end, (time T) and applying the modified Bellman equation and the law of motion for P_t , the relation between pending and executed shares, and the boundary conditions recursively, the optimal control can be derived as functions of the state variables that characterize the information that the investor must have to make his decision in each period. In particular, the optimal value function, $V_T(\dots)$, as a function of the two state variables P_{T-1} and W_T is given by,

$$V_T(P_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max\{(P_T - P_{T-1}), 0\} S_T]$$

Here, the remaining shares W_{T+1} must be zero since there is no choice but to execute all the remaining shares, W_T . We then have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, W_T) = E_T [\max\{(\theta W_T + \varepsilon_T), 0\} W_T]$$

Proposition 2. *The value function for the last but one time period is convex and can be written as,*

$$V_{T-1}(P_{T-2}, W_{T-1}) = \min_{\{S_{T-1}\}} [S_{T-1}\sigma_\varepsilon\psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon\psi\{\xi(W_{T-1} - S_{T-1})\}]$$

$$\text{Here, } \psi(u) = u + \phi(u)/\Phi(u), \quad \xi = \frac{\theta}{\sigma_\varepsilon},$$

Also, ϕ and Φ are the standard normal PDF and CDF, respectively.

Proof. See Appendix 7.2. □

Figure 3 illustrates the shape of some combinations of the distribution functions that we are working with. For the value function we have, the condition for convexity can be derived as $\theta > (3\sigma_\varepsilon/4)$.

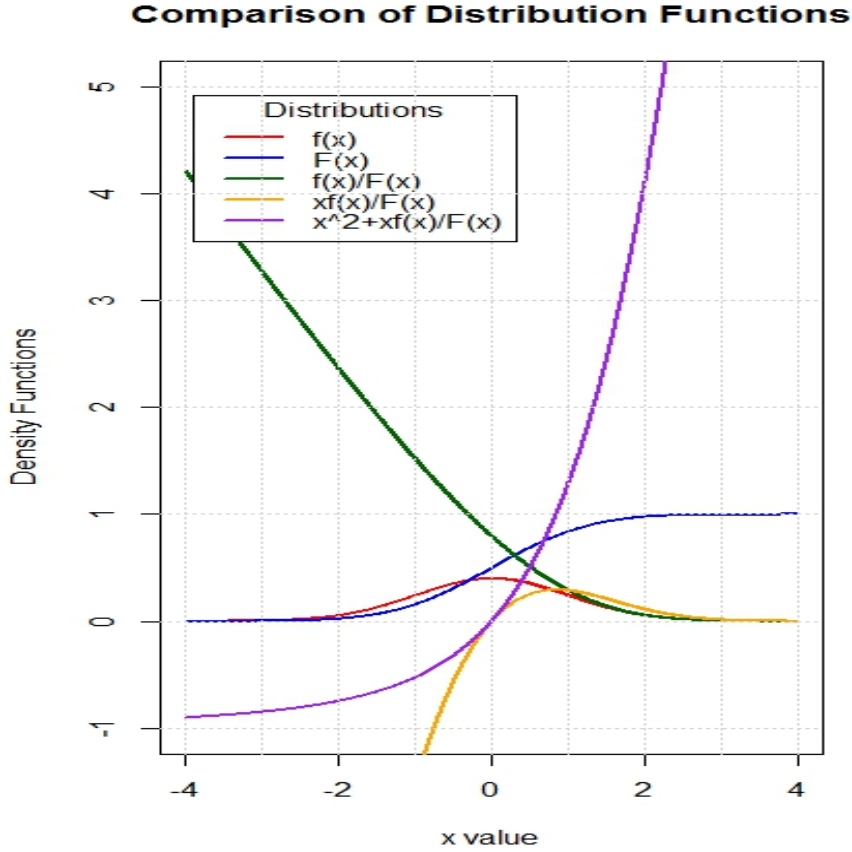


Figure 3: Convexity of Distribution Functions

Proposition 3. *The number of shares to be executed in each time period follows a linear law. $S_{T-1} =$*

$W_{T-1}/2 \dots S_{T-K-1} = W_{T-K-1}/(K+2)$ and the corresponding value functions are,

$$V_{T-K-1}(P_{T-K-2}, W_{T-K-1}) = \sigma_\varepsilon W_{T-K-1} \left[\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)} \right]$$

Proof. This is shown using induction in Appendix 7.3

□

We can see that a minimum exists at each stage. The simple solution follows from the linear rule where the price impact θS_t does not depend on either the prevailing price P_{t-1} or the size of the unexecuted order W_t and hence the price impact function is the same in each period and independent from one period to the next. (Bertsimas and Lo 1998) has a more detailed discussion. Supposing a closed form solution was absent, we could easily approximate the solution (numerically solved) using $S_{T-1} \approx \xi_0 + \xi_1 W_{T-1} + \xi_2 (W_{T-1})^2$. We can also set $S_{T-1} \approx \omega_1(W_{T-1})$ using a well behaved (continuous and differentiable) function, ω_1 . But the former approach is simpler and lends itself easily to numerical solutions that we will attempt in the more complex laws of motion to follow.

4.4.2 Complex Formulation

Incorporating the Complex Market Impact formulation from the earlier sections, the objective function and the Bellman equation can be modified as,

$$\begin{aligned} \min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] W_t \} \right] \\ \sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1} \\ P_t = P_{t-1} + \theta S_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0, \varepsilon_t \sim N(0, \sigma_\varepsilon^2) \end{aligned}$$

The Bellman equation then becomes,

$$V_t(P_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max\{(P_t - P_{t-1}), 0\} W_t + V_{t+1}(P_t, W_{t+1})]$$

The optimal value function, $V_T(\dots)$, as a function of the two state variables P_{T-1} and W_T is given by,

$$V_T(P_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max\{(P_T - P_{T-1}), 0\} W_T]$$

Here, the remaining shares W_{T+1} must be zero since there is no choice but to execute all the remaining shares, W_T . We then have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, W_T) = E_T[\max\{(\theta W_T + \varepsilon_T), 0\} W_T]$$

Proposition 4. *The number of shares to be executed in subsequent time periods and the corresponding value function are obtained by solving,*

$$\begin{aligned} W_{T-1} + \frac{\xi(W_{T-1} - S_{T-1})^2 \phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + (W_{T-1} - S_{T-1}) \left[\frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} \right]^2 &= \\ 2(W_{T-1} - S_{T-1}) + \frac{1}{\xi} \frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + \frac{\xi W_{T-1} S_{T-1} \phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + W_{T-1} \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right]^2 \end{aligned}$$

Proof. See Appendix 7.4. □

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, no longer applies here and we need numerical solutions at each stage. The complexity that gets included in this scenario when we consider the rest of the unexecuted program into the market impact function can be seen from this.

4.5 Law of Price Motion with Additional Source of Uncertainty

4.5.1 Simple Formulation

The law of price motion can be changed to include an additional source of uncertainty, X_t , which could represent changing market conditions or private information about the security. We assume that this state variable X_t , is serially-correlated and γ captures its sensitivity to the price movements. Incorporating this, the objective function and the Bellman equation become,

$$\begin{aligned} \min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] S_t\} \right] \\ \sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1} \\ P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0 \end{aligned}$$

$$X_t = \rho X_{t-1} + \eta_t, \quad \rho \in (-1, 1) \equiv \text{AR}(1) \text{ Process}$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N(0, \sigma_\eta^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t(P_{t-1}, X_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max \{(P_t - P_{t-1}), 0\} S_t + V_{t+1}(P_t, X_t, W_{t+1})]$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, X_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max \{(P_T - P_{T-1}), 0\} S_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T [\max \{(\theta W_T + \varepsilon_T + \gamma X_T), 0\} W_T]$$

Proposition 5. *The number of shares to be executed in each time period follows a linear law. $S_{T-1} = W_{T-1}/2 \quad \dots \quad S_{T-K-1} = W_{T-K-1}/(K+2)$ and the corresponding value function is*

$$V_{T-K-1}(P_{T-K-2}, X_{T-K-2}, W_{T-K-1}) = \frac{\theta}{(K+2)} W_{T-K-1}^2 + \alpha_{K+1} W_{T-K-1} + \beta W_{T-K-1} \frac{\phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}{\Phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}$$

$$\text{Here, } \alpha_{K+1} = \gamma \rho X_{T-K-2}, \quad \beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$$

Proof. See Appendix 7.5.

□

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, suffices even here, with a similar reasoning that follows from the independence of the price impact from either the prevailing price or the size of the unexecuted order.

4.5.2 Complex Formulation

Incorporating this additional source of uncertainty into the complex market impact formulation, the objective function and the Bellman equation become,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max [(P_t - P_{t-1}), 0] W_t \} \right]$$

$$\sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1}$$

$$P_t = P_{t-1} + \theta S_t + \gamma X_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0$$

$$X_t = \rho X_{t-1} + \eta_t, \rho \in (-1, 1) \equiv \text{AR}(1) \text{ Process}$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N(0, \sigma_\eta^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t(P_{t-1}, X_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max \{(P_t - P_{t-1}), 0\} W_t + V_{t+1}(P_t, X_t, W_{t+1})]$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, X_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max \{(P_T - P_{T-1}), 0\} W_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T [\max \{(\theta W_T + \varepsilon_T + \gamma X_T), 0\} W_T]$$

Proposition 6. *The number of shares to be executed in each time period and the corresponding value function are obtained by solving,*

$$\begin{aligned}
& \theta W_{T-1} + \beta W_{T-1} \left\{ \frac{\theta}{\beta} \left[-\frac{\left(\frac{\theta S_{T-1} + \alpha}{\beta} \right) \phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)} - \left\{ \frac{\phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)} \right\}^2 \right] \right\} \\
& - 2\theta (W_{T-1} - S_{T-1}) - \alpha + \beta \left\{ -\frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} \right. \\
& \left. + \frac{\theta (W_{T-1} - S_{T-1})}{\beta} \left[\frac{\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right) \phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} + \left\{ \frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} \right\}^2 \right] \right\} = 0
\end{aligned}$$

Proof. See Appendix 7.6. □

The simple rule established earlier, $S_{T-1} = W_{T-1}/2$, no longer suffices here and we need numerical solutions at each stage of the recursion.

4.6 Linear Percentage Law of Price Motion

4.6.1 Simple Formulation

A law of motion based on an arithmetic random walk has a positive probability of negative prices and it also implies that the Market Impact has a permanent effect on the prices. The other issue is that Market Impact as a percentage of the execution price is a decreasing function of the price level, which is counter-factual. Hence we let the execution price be comprised of two components, a no-impact price \tilde{P}_t , and the price impact Δ_t .

$$P_t = \tilde{P}_t + \Delta_t$$

The no impact price is the price that would prevail in the absence of any market impact. An observable proxy for this is the mid-point of the bid/offer spread. This is the natural price process and we set it to be a Geometric Brownian Motion.

$$\tilde{P}_t = \tilde{P}_{t-1} e^{B_t}$$

$$B_t \sim N(\mu_B, \sigma_B^2) \equiv \text{IID (Independent Identically Distributed) normal random variable}$$

The price impact Δ_t captures the effect of trade size on the transaction price including the portion of the bid/offer spread. As a percentage of the no-impact price \tilde{P}_t , it is a linear function of the trade size S_t and

X_t where as before, X_t is a proxy for private information or market conditions. The parameters θ and γ measure the sensitivity of price impact to trade size and market conditions or private information.

$$\Delta_t = (\theta S_t + \gamma X_t) \tilde{P}_t$$

$$X_t = \rho X_{t-1} + \eta_t, \quad \rho \in (-1, 1)$$

$\eta_t \sim N(0, \sigma_\eta^2) \equiv$ Zero Mean IID (Independent Identically Distributed) random shock or white noise

The optimization problem and Bellman equation can be written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max[(P_t - P_{t-1}), 0] S_t \} \right]$$

$$\sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1}$$

$$V_t(P_{t-1}, X_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max\{(P_t - P_{t-1}), 0\} S_t + V_{t+1}(P_t, X_t, W_{t+1})]$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, X_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max\{(P_T - P_{T-1}), 0\} S_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T \left[\max \left\{ \left(\tilde{P}_T (1 + \theta W_T + \gamma X_T) - P_{T-1} \right), 0 \right\} W_T \right]$$

This involves a normal log-normal mixture and solutions are known for handling this distribution under certain circumstances (Clark 1973; Tauchen and Pitts 1983 ; Yang 2008).

Proposition 7. *The value function is of the form, $E[Y_2 | Y_2 > 0]$ where,*

$Y_2 = \left(\tilde{P}_{T-1} W_T e^{B_T} + \theta W_T^2 \tilde{P}_{T-1} e^{B_T} + \gamma \rho X_{T-1} \tilde{P}_{T-1} W_T e^{B_T} + \gamma \tilde{P}_{T-1} W_T e^{B_T} \eta_T - W_T P_{T-1} \right)$. *This can be simplified further to,*

$$E[(e^X Y + k) | (e^X Y + k) > 0]$$

$$\begin{aligned}
= & k + e^{(\mu_X + \frac{1}{2}\sigma_X^2)} \left[\left\{ \frac{\Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \left\{ \mu_Y \left[\Phi\left(-\left[\frac{k + \mu_Y}{\sigma_Y}\right]\right) - \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) \right] - \frac{\sigma_Y}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{k + \mu_Y}{\sigma_Y}\right)^2} - e^{-\frac{1}{2}\left(\frac{\mu_Y}{\sigma_Y}\right)^2} \right] \right\} \right. \\
& \left. + \left\{ \frac{1 - \Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{1 - \Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \left\{ \mu_Y \left[1 - \Phi\left(-\left[\frac{k + \mu_Y}{\sigma_Y}\right]\right) \right] + \frac{\sigma_Y}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{k + \mu_Y}{\sigma_Y}\right)^2} \right] \right\} \right]
\end{aligned}$$

Here, $X \sim N(\mu_X, \sigma_X^2)$; $Y \sim N(\mu_Y, \sigma_Y^2)$; X and Y are independent. Also, $k < 0$

Proof. See Appendix 7.7.

□

We can also use numerical techniques (Miranda and Fackler 2002) or approximations to the error function (Chiani, Dardari and Simon 2003). An alternative approach is to use least squares to approximate the conditional expectation as a function of the state variables at each stage, similar to the methodology used in pricing american options (Longstaff and Schwartz 2001).

4.6.2 Complex Formulation

The optimization problem and Bellman equation for the complex case can be written as,

$$\begin{aligned}
& \min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{\max[(P_t - P_{t-1}), 0] W_t\} \right] \\
& \sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1} \\
& V_t(P_{t-1}, X_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max\{(P_t - P_{t-1}), 0\} W_t + V_{t+1}(P_t, X_t, W_{t+1})]
\end{aligned}$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, X_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max\{(P_T - P_{T-1}), 0\} W_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T can be arrived similar to the simple formulation in Proposition 7.

4.7 Introducing Liquidity Constraints

4.7.1 Simple Formulation

A practical limitation that arises when trading is the extent of liquidity that is available at any point in time. This becomes a restriction on the amount of shares tradable in any given interval. Volume can be observed and estimated with a reasonable degree of accuracy. Hence, any measure linking volume to trading costs would be a very practical device. There is a voluminous literature that derives theoretical models and looks at the empirical relationship between volume and prices. (Karpoff 1986 and 1987; Gallant, Rossi and Tauchen 1992; Campbell, Grossman and Wang 1993; Wang 1994). We fit a specification similar to the one in Campbell et al wherein the price movements can arise due to changes in future cash flows and investor preferences or the risk aversion. The intuition for this would be that a low return due to a price drop could be caused by an increase in the risk aversion or bad news about future cash flows. Changes in risk aversion cause trading volume to increase while news that is public will already have been impounded in the price and hence will not cause additional trading. Low returns followed by high volume are due to increased risk aversion while low returns and low volume are due to public knowledge of a low level of expectation of future returns. As risk aversion increases, the group of investor still willing to hold the stock require a greater return leading to higher future expected returns. Bad news about future cash flows leads to lower expected returns. This is captured as an inverse relation between auto-correlation of returns and trading volume. The simplification we employ combines the two sources of price changes into one, since what can be observed is only the price return. We note that this can be viewed as an extension of the law of price motion with an additional source of uncertainty. Here, O_t is the total volume traded (market volume) in the interval t . The co-efficient α can be positive or negative, γ is positive and θ continues to be positive.

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max [(P_t - P_{t-1}), 0] S_t \} \right]$$

$$\sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1}$$

$$P_t = (\alpha + 1) P_{t-1} + \theta S_t P_{t-1} - \gamma (O_t - S_t) P_{t-1} + \varepsilon_t, O_t \geq S_t, \beta, \theta > 0, \alpha \in (-\infty, \infty), E[\varepsilon_t | S_t, P_{t-1}] = 0$$

$$O_t = \rho O_{t-1} + \eta_t, \rho \in (-1, 1) \equiv \text{AR}(1) \text{ Process}$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N(0, \sigma_\eta^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t(P_{t-1}, O_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max \{ (P_t - P_{t-1}), 0 \} S_t + V_{t+1}(P_t, O_t, W_{t+1})]$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, O_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max \{(P_T - P_{T-1}), 0\} S_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T as,

Proposition 8. *The value functions are of the form, $E[Y|Y > 0]$ where,*

$Y = (\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T)$. For the last and last but one time periods, these can be simplified further to,

$$V_T(P_{T-1}, O_{T-1}, W_T) = \left(\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) W_T \psi(\xi W_T), \quad \xi W_T = \left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)$$

and

$$\begin{aligned} V_{T-1}(P_{T-2}, O_{T-2}, W_{T-1}) = & \min_{\{S_{T-1}\}} E_{T-1} \left\{ S_{T-1} \left(\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \right. \\ & \left[\left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) + \frac{\phi \left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \\ & + (W_{T-1} - S_{T-1}) \left(\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \\ & \left[\left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right. \\ & \left. + \frac{\phi \left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \left. \right\} \end{aligned}$$

Here, $\beta = \theta + \gamma$,

Proof. See Appendix 7.8.

□

This requires numerical solutions at each stage of the recursion. A point worth noting is that the simple rule from the earlier linear cases, where the price impact is independent of both the prevailing price and the size of the unexecuted order, no longer applies here. The necessity of having to work with complicated

expressions of the sort above, highlights to us the inherent difficulty of making predictions in a complex social system and also that our approach to estimating Market Impact provides a realistic platform upon which further complications, such as working with joint distributions of volume and price, can be built. A key takeaway from this result is that volume can have counter intuitive effects on the trading costs.

4.7.2 Complex Formulation

The optimization problem and Bellman equation for the complex case can be written as,

$$\min_{\{S_t\}} E_1 \left[\sum_{t=1}^T \{ \max [(P_t - P_{t-1}), 0] W_t \} \right]$$

$$\sum_{t=1}^T S_t = \bar{S}, S_t \geq 0, W_1 = \bar{S}, W_{T+1} = 0, W_t = W_{t-1} - S_{t-1}$$

$$P_t = (\alpha + 1) P_{t-1} + \theta S_t P_{t-1} - \gamma (O_t - S_t) P_{t-1} + \varepsilon_t, O_t \geq S_t, \beta, \theta > 0, \alpha \in (-\infty, \infty), E[\varepsilon_t | S_t, P_{t-1}] = 0$$

$$O_t = \rho O_{t-1} + \eta_t, \rho \in (-1, 1) \equiv \text{AR}(1) \text{ Process}$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N(0, \sigma_\eta^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$V_t(P_{t-1}, O_{t-1}, W_t) = \min_{\{S_t\}} E_t [\max \{(P_t - P_{t-1}), 0\} S_t + V_{t+1}(P_t, O_t, W_{t+1})]$$

By starting at the end, (time T) we have,

$$V_T(P_{T-1}, O_{T-1}, W_T) = \min_{\{S_T\}} E_T [\max \{(P_T - P_{T-1}), 0\} W_T]$$

Since W_{T+1} is zero, we have the optimal trade size, $S_T^* = W_T$ and an expression for V_T can be arrived similar to the simple formulation in Proposition 8.

4.8 Trading Costs and Price Spread Sandwich

4.8.1 Simple Formulation

Another useful tool from a trading perspective would be a measure that connects trading costs to the spread, which can be observed. (Roll 1984 and Stoll 1989) connect the stock price changes to the bid-offer spread. The spread is determined due to order processing costs, adverse information or inventory holdings costs. The covariance of price changes are related to the covariance of the changes in spread and proportional to

the square of the spread, assuming constant spread. A modification with time varying spread can be easily accommodated in the specifications above with an additional source of uncertainty or the linear percentage law of motion. Here, Q_t is the spread at any point in time.

$$P_t = P_{t-1} + \theta S_t + \gamma Q_t + \varepsilon_t, \theta > 0, E[\varepsilon_t | S_t, P_{t-1}] = 0$$

$$Q_t = \rho Q_{t-1} + \eta_t, \rho \in (-1, 1) \equiv \text{AR}(1) \text{ Process}$$

$$\varepsilon_t \sim N(0, \sigma_\varepsilon^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

$$\eta_t \sim N(0, \sigma_\eta^2) \equiv \text{Zero Mean IID (Independent Identically Distributed) random shock or white noise}$$

4.8.2 Complex Formulation

This would be analogous to the case in 4.8.1.

4.9 Trading Costs Attribution

The following diagram (Figure 4) illustrates the distribution of trading costs based on the above attribution methodology. (Kashyap 2015b, 2016) are empirical examples of applying the above methodology to recent market events, wherein, Mincer Zarnowit type regressions (Mincer and Zarnowitz 1969) are run to establish the accuracy of the estimates. These studies demonstrate the effectiveness of this approach in helping us better understand and analyze real life trading situations.

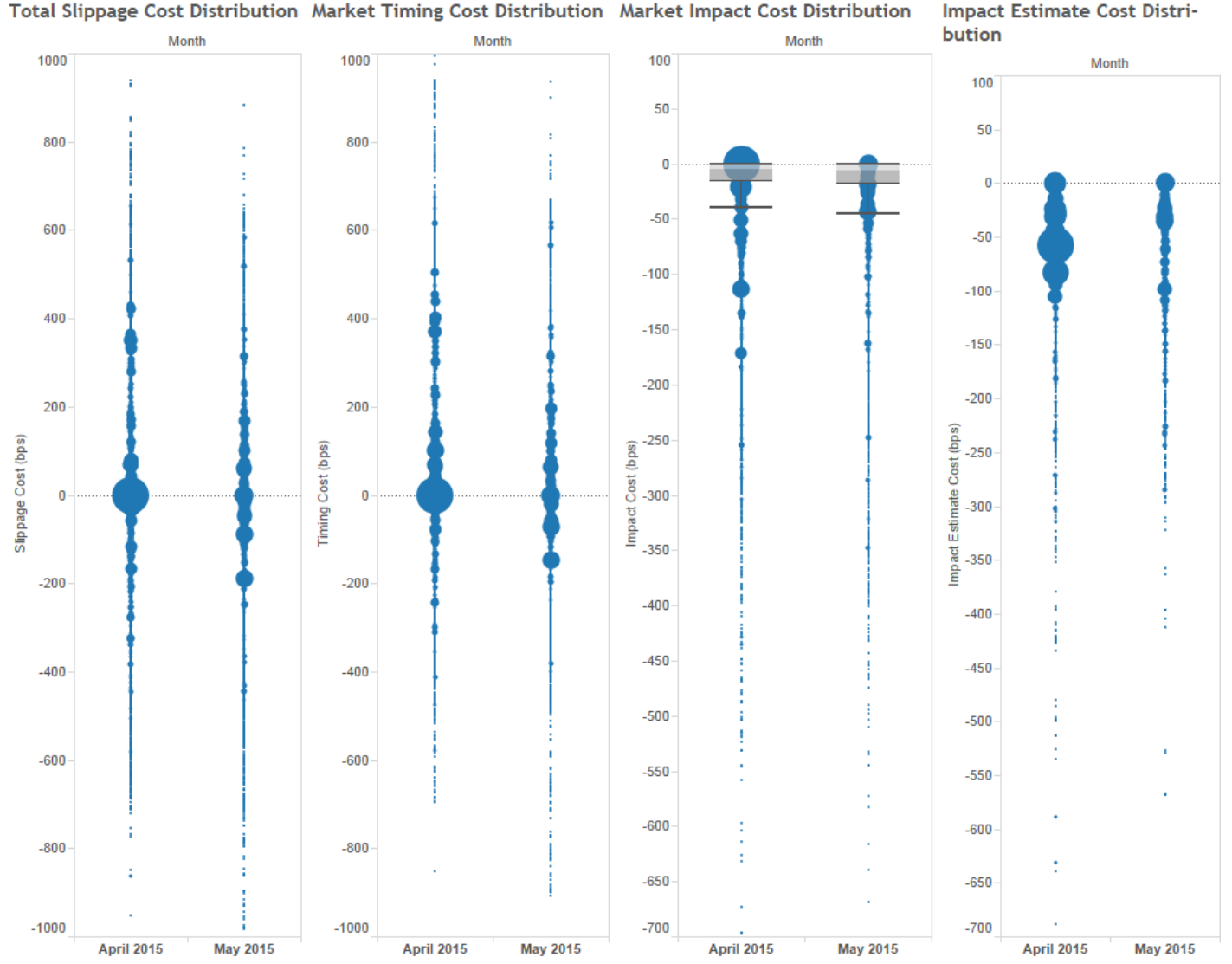


Figure 4: Trading Cost Distributions

5 Conclusions and Possibilities for Future Research

We have developed a trading cost model using dynamic programming that splits the overall price move into the market impact and timing components. To ensure that this can be used under different situations, we build upon the benchmark case and introduce more complex formulations of the law of price motion, including a scenario that has multiple sources of uncertainty and consider liquidity or volume constraints. Relating the trading costs to the spread is also easily accomplished. Key improvements to the model and methodology would stem from adding cases where volume and prices are not assumed to be independent. Distributions of prices that are not normal and factor the downward skew in prices might also provide more

realistic estimates. Our model takes the prices process as exogenous, interesting continuations can extend the separation of impact and timing to models of the limit order book that endogenously consider the evolution of prices. Again we stress the better insights and understanding that results from using simpler models, but the particulars of the securities being considered might prompt experimenting with some of the more esoteric extensions. We have looked at only discrete time formulations, extensions to continuous time might show interesting theoretical behavior.

While ambitions to produce a normative work on trading costs would be commendable, the present state of affairs in social science modeling would perhaps mean that our efforts would fall far short of a satisfactory solution. Though our work considers many elements of a trading system at face value, it cannot be termed positive, and if it ends up producing useful finance (McGoun 2003), its purpose is well served.

6 Notes and References

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7 Appendix

7.1 Proof of Proposition 1

Proof. For the simple formulation, with one interval and two market participants (one is the buyer, the other is the seller) and remembering that without two participants we do not have market or a trade,

For the buyer we have,

$$\text{Market Impact} = \{\max[(P_t - P_{t-1}), 0] S_t\}$$

$$\begin{aligned}
\text{Market Timing} &= \text{Implementation Shortfall} - \text{Market Impact} \\
&= (S_t P_t) - S_t P_0 - \{\max[(P_t - P_{t-1}), 0] S_t\}
\end{aligned}$$

Similarly we have for the seller,

$$\text{Market Impact} = \{\max[(P_{t-1} - P_t), 0] S_t\}$$

$$\begin{aligned}
\text{Market Timing} &= -\text{Implementation Shortfall} - \text{Market Impact} \\
&= S_t P_0 - (S_t P_t) - \{\max[(P_{t-1} - P_t), 0] S_t\}
\end{aligned}$$

If $(P_t > P_{t-1})$,

For the buyer,

$$\text{Market Impact} = (P_t - P_{t-1}) S_t$$

$$\text{Market Timing} = (S_t P_t) - S_t P_0 - (P_t - P_{t-1}) S_t = (P_{t-1} - P_0) S_t$$

For the seller,

$$\text{Market Impact} = 0$$

$$\text{Market Timing} = S_t P_0 - (S_t P_t) - 0 = (P_0 - P_t) S_t$$

Sum of the impact and timing across both the participants,

$$\text{Total Market Impact} = (P_t - P_{t-1}) S_t$$

$$\text{Total Market Timing} = (P_{t-1} - P_0) S_t + (P_0 - P_t) S_t = (P_{t-1} - P_t) S_t$$

$$\text{Total Market Impact} + \text{Total Market Timing} = 0$$

The other cases including multiple intervals, multiple participants and the complex formulation are easily shown. □

7.2 Proof of Proposition 2

Proof. We have from the value function for the last time period.

$$V_T(P_{T-1}, W_T) = E_T [\max \{(\theta W_T + \varepsilon_T), 0\} W_T]$$

$$V_T(P_{T-1}, W_T) = E_T [\max \{(\theta W_T^2 + W_T \varepsilon_T), 0\}]$$

$$V_T(P_{T-1}, W_T) = E_T [(\theta W_T^2 + W_T \varepsilon_T) | (\theta W_T^2 + W_T \varepsilon_T) > 0]$$

$$\{\cdot \cdot E[\max(X, c)] = E[X | X > c] Pr[X > c] + E[c | X \leq c] Pr[X \leq c] \}$$

This is of the form, $E[Y | Y > 0]$ where, $Y = (\theta W_T^2 + W_T \varepsilon_T)$. We then need to calculate,

$$E_T \left[(\theta W_T^2 + W_T \sigma_\varepsilon Z) | Z > \left(-\frac{\theta W_T}{\sigma_\varepsilon} \right) \right], \text{ where } Z \sim N(0, 1)$$

$$[\cdot \cdot Y \sim N(\theta W_T^2, W_T^2 \sigma_\varepsilon^2) \equiv Y \sim N(\mu, \sigma^2) \Rightarrow Y = \mu + \sigma Z; Y > 0 \Rightarrow Z > -\mu/\sigma]$$

We have for every standard normal distribution, Z , and for every u , $Pr[Z > -u] = Pr[Z < u] = \Phi(u)$.

Here, ϕ and Φ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E[Z | Z > -u] &= \frac{1}{\Phi(u)} \left[\int_{-u}^{\infty} t \phi(t) dt \right] \\ &= \frac{1}{\Phi(u)} [-\phi(t)]_{-u}^{\infty} = \frac{\phi(u)}{\Phi(u)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E[Y | Y > 0] &= \mu + \sigma E \left[Z | Z > \left(-\frac{\mu}{\sigma} \right) \right] \\ &= \mu + \frac{\sigma \phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \end{aligned}$$

Setting, $\psi(u) = u + \phi(u)/\Phi(u)$,

$$E[Y | Y > 0] = \sigma \psi(\mu/\sigma)$$

$$\begin{aligned} V_T(P_{T-1}, W_T) &= E_T [(\theta W_T^2 + W_T \varepsilon_T) | (\theta W_T^2 + W_T \varepsilon_T) > 0] \\ &= W_T \sigma_\varepsilon \left[\frac{\theta W_T}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)} \right] = \sigma_\varepsilon W_T \psi(\xi W_T), \quad \xi = \frac{\theta}{\sigma_\varepsilon} \end{aligned}$$

In the next to last period, $T - 1$, the Bellman equation is,

$$\begin{aligned}
V_{T-1}(P_{T-2}, W_{T-1}) &= \min_{\{S_{T-1}\}} E_{T-1} [\max \{(P_{T-1} - P_{T-2}), 0\} S_{T-1} + V_T(P_{T-1}, W_T)] \\
&= \min_{\{S_{T-1}\}} E_{T-1} [\max \{(\theta S_{T-1} + \varepsilon_{T-1}), 0\} S_{T-1} + V_T(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1}, W_{T-1} - S_{T-1})] \\
&= \min_{\{S_{T-1}\}} \left\{ S_{T-1} \sigma_\varepsilon \left[\frac{\theta S_{T-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)} \right] + (W_{T-1} - S_{T-1}) \sigma_\varepsilon \left[\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)} \right] \right\} \\
&= \min_{\{S_{T-1}\}} [S_{T-1} \sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1}) \sigma_\varepsilon \psi\{\xi (W_{T-1} - S_{T-1})\}]
\end{aligned}$$

We show this to be a convex function with a unique minimum. Let us start with,

$$\begin{aligned}
G(x) &= \frac{\phi(x)}{\Phi(x)} \quad | \forall x > 0 \\
\frac{\partial G(x)}{\partial x} &= \frac{\phi'(x)}{\Phi(x)} - \left[\frac{\phi(x)}{\Phi(x)} \right]^2 \\
\frac{\partial G(x)}{\partial x} &= -\frac{x\phi(x)}{\Phi(x)} - \left[\frac{\phi(x)}{\Phi(x)} \right]^2 \\
\left[\because \frac{\partial \phi(x)}{\partial (x)} = -x\phi(x) ; \frac{\partial \Phi(x)}{\partial (x)} = \phi(x) \right] \\
\frac{\partial G(x)}{\partial x} &= \frac{\phi(x)}{\Phi(x)} \left[-x - \frac{\phi(x)}{\Phi(x)} \right] < 0 \quad | \forall x > 0 \\
\frac{\partial^2 G(x)}{\partial x^2} &= -\frac{\phi(x)}{\Phi(x)} - x \left\{ -\frac{x\phi(x)}{\Phi(x)} - \left[\frac{\phi(x)}{\Phi(x)} \right]^2 \right\} - \frac{2\phi(x)\phi'(x)}{\Phi^2(x)} + \frac{2\phi^3(x)}{\Phi^3(x)} \\
&= \frac{-\phi(x)\Phi^2(x) + x^2\phi(x)\Phi^2(x) + x\phi^2(x)\Phi(x) + 2x\phi^2(x)\Phi(x) + 2\phi^3(x)}{\Phi^3(x)} \\
&= \frac{\phi(x)\{-\Phi^2(x) + x^2\Phi^2(x) + 3x\phi(x)\Phi(x) + 2\phi^2(x)\}}{\Phi^3(x)}
\end{aligned}$$

Consider the following, $\forall x > 0$

$$\text{Let } K(x) = 3x\phi(x)\Phi(x) + 2\phi^2(x) + (x^2 - 1)\Phi^2(x)$$

For $x \geq 1, K(x) > 0$. Also,

$$K(0) = \frac{1}{\pi} - \frac{1}{4} > 0$$

$$\left[\because \phi(x) = \frac{e^{-\frac{1}{2}x^2}}{\sqrt{2\pi}} ; \phi(0) = \frac{1}{\sqrt{2\pi}} ; \Phi(x) = \frac{1}{2} \left\{ 1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) \right\} ; \Phi(0) = \frac{1}{2} \right]$$

For $x \in (0, 1)$,

$$K'(x) = 3\phi(x)\Phi(x) - 3x^2\phi(x)\Phi(x) + 3x\phi^2(x) - 4x\phi^2(x) + (x^2 - 1)2\Phi(x)\phi(x) + 2x\Phi^2(x)$$

$$K'(x) = x\{2\Phi^2(x) - \phi^2(x)\} + \{1 - x^2\}\phi(x)\Phi(x) \geq xL(x)$$

where, $L(x) = \{2\Phi^2(x) - \phi^2(x)\}$. Further, $\Phi(x) \geq \frac{1}{2}$ and $\phi^2(x) \leq \frac{1}{2\pi} \Rightarrow L(x) \geq \frac{1}{2} - \frac{1}{2\pi} > 0$. $K'(x) > 0 \Rightarrow K(x)$ is increasing. Hence, $K(x) > K(0) > 0 \mid \forall x \in (0, 1)$. This gives, $K(x) > 0$ and $\frac{\partial^2 G(x)}{\partial x^2} > 0 \mid \forall x \in (0, \infty)$. It is worth noting the following asymptotic properties

$$\left[\because \lim_{x \rightarrow 0+} \frac{\partial^2 G(x)}{\partial x^2} > 0 ; \lim_{x \rightarrow \infty} \frac{\partial^2 G(x)}{\partial x^2} = 0 ; \frac{\partial G(x)}{\partial x} < 0 \mid \forall x > 0 \right]$$

Next we show that $f(a - x)$ is convex, given $f''(x) > 0 ; x > 0$

Let $y = a - x$

$$\frac{\partial f(y)}{\partial x} = \frac{\partial f(y)}{\partial y} \frac{\partial(a - x)}{\partial x} ; a > x$$

$$= (-1) f'(y)$$

$$\frac{\partial^2 f(y)}{\partial x^2} = (-1) \frac{\partial f'(y)}{\partial y} \frac{\partial(a - x)}{\partial x}$$

$$= f''(y)$$

$$> 0 [\because f''(y) > 0 \mid \forall y > 0]$$

We can similarly show that $f(bx)$ is convex if $f(x)$ is convex (in our case, $b > 0$, but the result holds $\forall b$).

Next we derive conditions when $x^2 + xf(x)$ is convex, given $f''(x) > 0$; $x > 0$

$$\text{Let } g(x) = x^2 + xf(x)$$

$$g'(x) = 2x + xf'(x) + f(x)$$

$$g''(x) = 2 + xf''(x) + 2f'(x)$$

$$g''(x) > 0 \text{ if } f'(x) > 0 \text{ or if } 2 + xf''(x) > |2f'(x)|$$

Finally,

$$\text{Let } Q(x) = x^2 + x \frac{\phi(x)}{\Phi(x)}$$

$$\frac{\partial Q(x)}{\partial x} = 2x + x \left[-\frac{x\phi(x)}{\Phi(x)} - \left\{ \frac{\phi(x)}{\Phi(x)} \right\}^2 \right] + \frac{\phi(x)}{\Phi(x)}$$

$$\left. \frac{\partial Q(x)}{\partial x} \right|_{x=0} = \frac{\phi(0)}{\Phi(0)} > 0$$

$$\begin{aligned} \frac{\partial^2 Q(x)}{\partial x^2} &= 2 + x\phi(x) \left[\frac{-\Phi^2(x) + x^2\Phi^2(x) + 3x\phi(x)\Phi(x) + 2\phi^2(x)}{\Phi^3(x)} \right] + 2 \left[-x \frac{\phi(x)}{\Phi(x)} - \left\{ \frac{\phi(x)}{\Phi(x)} \right\}^2 \right] \\ &= 2 + \phi(x) \left[\frac{x^3\Phi^2(x) + 3x^2\phi(x)\Phi(x) + 2x\phi^2(x) - 3x\Phi^2(x) - 2\phi(x)\Phi(x)}{\Phi^3(x)} \right] \\ &= \left[\frac{2\Phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) + 2x\phi^3(x) - 3x\Phi^2(x)\phi(x) - 2\phi^2(x)\Phi(x)}{\Phi^3(x)} \right] \end{aligned}$$

$$\text{Let, } K(x) = 2\Phi^3(x) + 2x\phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) - 3x\Phi^2(x)\phi(x) - 2\phi^2(x)\Phi(x)$$

We need to show that $K(x) > 0 \forall x > 0$. First we note that,

$$K(0) = \frac{1}{4} - \frac{1}{2\pi} > 0$$

We can write this as,

$$K(x) = 2x\phi^3(x) + x^3\Phi^2(x)\phi(x) + 3x^2\phi^2(x)\Phi(x) + \Phi(x) [2\Phi^2(x) - 3x\Phi(x)\phi(x) - 2\phi^2(x)]$$

We then need to show,

$$L(x) = [2\Phi^2(x) - 3x\Phi(x)\phi(x) - 2\phi^2(x)] > 0 \mid \forall x > 0$$

$$L(0) = [2\Phi^2(0) - 2\phi^2(0)] = \frac{1}{2} - \frac{1}{\pi} > 0$$

$$\begin{aligned} \frac{\partial L(x)}{\partial x} &= 4\Phi(x)\phi(x) - 3\Phi(x)\phi(x) - 3x\phi^2(x) + 3x^2\Phi(x)\phi(x) + 4x\phi^2(x) \\ &= \Phi(x)\phi(x) + 3x^2\Phi(x)\phi(x) + x\phi^2(x) > 0 \mid \forall x \geq 0 \end{aligned}$$

Therefore $L(x)$ is an increasing function on the interval $[0, \infty)$. Its minimum must be at $L(0) > 0$, proving $L(x) > 0$ and $\frac{\partial^2 Q(x)}{\partial x^2} > 0 \mid \forall x \in (0, \infty)$. It is worth noting the following asymptotic properties and the graphical results shown below,

$$\left[\because \lim_{x \rightarrow 0+} \frac{\partial^2 Q(x)}{\partial x^2} > 0; \lim_{x \rightarrow \infty} \frac{\partial^2 Q(x)}{\partial x^2} > 0; \frac{\partial Q(x)}{\partial x} > 0 \mid \forall x > 0 \right]$$

□

7.3 Proof of Proposition 3

Proof. Consider,

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) &= \min_{\{S_{T-1}\}} [S_{T-1}\sigma_\varepsilon\psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon\psi\{\xi(W_{T-1} - S_{T-1})\}] \\ \text{Here, } \psi(u) &= u + \phi(u)/\Phi(u), \quad \xi = \frac{\theta}{\sigma_\varepsilon}, \end{aligned}$$

First Order Conditions (FOC) give,

$$\frac{\partial}{\partial S_{T-1}} [S_{T-1}\sigma_\varepsilon\psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1})\sigma_\varepsilon\psi\{\xi(W_{T-1} - S_{T-1})\}] = 0$$

$$\xi S_{T-1}\psi'(\xi S_{T-1}) + \psi(\xi S_{T-1}) - \xi(W_{T-1} - S_{T-1})\psi'(\xi\{W_{T-1} - S_{T-1}\}) - \psi(\xi\{W_{T-1} - S_{T-1}\}) = 0$$

$$\begin{aligned} &\xi^2 S_{T-1} \left\{ 1 - \frac{\xi S_{T-1}\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} - \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right]^2 \right\} + \left\{ \xi S_{T-1} + \frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right\} \\ &+ \xi^2 (W_{T-1} - S_{T-1}) \left\{ -1 + \frac{\xi (W_{T-1} - S_{T-1})\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} + \left[\frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} \right]^2 \right\} \\ &\quad - \left\{ \xi\{W_{T-1} - S_{T-1}\} + \frac{\phi(\xi\{W_{T-1} - S_{T-1}\})}{\Phi(\xi\{W_{T-1} - S_{T-1}\})} \right\} = 0 \end{aligned}$$

$$\left[\because \psi'(u) = 1 - \frac{u\phi(u)}{\Phi(u)} - \left[\frac{\phi(u)}{\Phi(u)} \right]^2 \text{ and } \psi(u) = u + \frac{\phi(u)}{\Phi(u)} \right]$$

$$\begin{aligned} S_{T-1} + \frac{1}{\xi} S_{T-1} + \frac{1}{\xi^2} \frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + \frac{\xi (W_{T-1} - S_{T-1})^2 \phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \\ + (W_{T-1} - S_{T-1}) \left[\frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \right]^2 = \\ (W_{T-1} - S_{T-1}) + \frac{1}{\xi} \{W_{T-1} - S_{T-1}\} + \frac{1}{\xi^2} \frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \\ + \frac{\xi S_{T-1}^2 \phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + S_{T-1} \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right]^2 \end{aligned}$$

Setting $S_{T-1} = W_{T-1}/2$ gives $RHS = LHS$. We have,

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) &= S_{T-1} \sigma_\varepsilon \left[\frac{\theta S_{T-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)} \right] \\ &+ (W_{T-1} - S_{T-1}) \sigma_\varepsilon \left[\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)} \right] \\ V_{T-1}(P_{T-2}, W_{T-1}) &= W_{T-1} \sigma_\varepsilon \left[\frac{\theta W_{T-1}}{2\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_{T-1}}{2\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-1}}{2\sigma_\varepsilon}\right)} \right] \end{aligned}$$

Absent closed form solutions, numerical techniques using $\xi_1 > 0$ can be tried. We can also set $S_{T-1} \approx \omega_1(W_{T-1})$ using a well behaved (continuous and differentiable) function, ω_1 . But the former approach is simpler and lends itself easily to numerical solutions that we will attempt in the more complex laws of motion to follow.

$$S_{T-1} \approx \xi_1 W_{T-1}$$

or with additional terms as,

$$S_{T-1} \approx \xi_0 + \xi_1 W_{T-1} + \xi_2 (W_{T-1})^2$$

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) &= [\sigma_\varepsilon \xi_1 W_{T-1} \psi(\xi \xi_1 W_{T-1}) + \{W_{T-1} - \xi_1 W_{T-1}\} \sigma_\varepsilon \psi\{\xi (W_{T-1} - \xi_1 W_{T-1})\}] \\ &= \sigma_\varepsilon W_{T-1} [\xi_1 \psi(\xi \xi_1 W_{T-1}) + (1 - \xi_1) \psi\{\xi W_{T-1} (1 - \xi_1)\}] \\ &= \psi_1(W_{T-1}) \text{ , here, } \psi_1 \text{ is a convex function.} \end{aligned}$$

Continuing the recursion,

$$V_{T-2}(P_{T-3}, W_{T-2}) = \min_{\{S_{T-2}\}} E_{T-2} [\max\{(P_{T-2} - P_{T-3}), 0\} S_{T-2} + V_{T-1}(P_{T-2}, W_{T-1})]$$

$$\begin{aligned}
&= \min_{\{S_{T-2}\}} E_{T-2} [\max \{(\theta S_{T-2} + \varepsilon_{T-2}), 0\} S_{T-2} + V_{T-1}(P_{T-3} + \theta S_{T-2} + \varepsilon_{T-3}, W_{T-2} - S_{T-2})] \\
&= \min_{\{S_{T-2}\}} \left[S_{T-2} \sigma_\varepsilon \left\{ \frac{\theta S_{T-2}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)} \right\} + (W_{T-2} - S_{T-2}) \sigma_\varepsilon \left\{ \frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon} + \frac{\phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]} \right\} \right]
\end{aligned}$$

First Order Conditions (FOC) give,

$$\begin{aligned}
&3\theta S_{T-2} - \theta W_{T-2} + \frac{\sigma_\varepsilon \phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)} - \frac{\theta^2 S_{T-2}^2 \phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\sigma_\varepsilon \Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)} - \theta S_{T-2} \left[\frac{\phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-2}}{\sigma_\varepsilon}\right)} \right]^2 \\
&- \frac{\sigma_\varepsilon \phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]} + \frac{\theta^2 (W_{T-2} - S_{T-2})^2 \phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]}{4\sigma_\varepsilon \Phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]} + \frac{\theta (W_{T-2} - S_{T-2})}{2} \left\{ \frac{\phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]}{\Phi\left[\frac{\theta(W_{T-2} - S_{T-2})}{2\sigma_\varepsilon}\right]} \right\}^2 = 0
\end{aligned}$$

This gives, $S_{T-2} = W_{T-2}/3$ and the corresponding value function as,

$$\begin{aligned}
V_{T-2}(P_{T-3}, W_{T-2}) &= \frac{W_{T-2}}{3} \sigma_\varepsilon \left\{ \frac{\theta}{\sigma_\varepsilon} \frac{W_{T-2}}{3} + \frac{\phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-2}}{3}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-2}}{3}\right)} \right\} + \left(\frac{2W_{T-2}}{3} \right) \sigma_\varepsilon \left\{ \frac{\theta}{2\sigma_\varepsilon} \frac{2W_{T-2}}{3} + \frac{\phi\left[\frac{\theta}{2\sigma_\varepsilon} \frac{2W_{T-2}}{3}\right]}{\Phi\left[\frac{\theta}{2\sigma_\varepsilon} \frac{2W_{T-2}}{3}\right]} \right\} \\
V_{T-2}(P_{T-3}, W_{T-2}) &= \sigma_\varepsilon W_{T-2} \left[\frac{\theta W_{T-2}}{3\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_{T-2}}{3\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-2}}{3\sigma_\varepsilon}\right)} \right]
\end{aligned}$$

We show the general case using induction. Let the value function hold for $T - K$

$$V_{T-K}(P_{T-K-1}, W_{T-K}) = \sigma_\varepsilon W_{T-K} \left[\frac{\theta W_{T-K}}{(K+1)\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_{T-K}}{(K+1)\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_{T-K}}{(K+1)\sigma_\varepsilon}\right)} \right]$$

Continuing the recursion,

$$\begin{aligned}
V_{T-K-1}(P_{T-K-2}, W_{T-K-1}) &= \min_{\{S_{T-K-1}\}} E_{T-K-1} [\max \{(P_{T-K-1} - P_{T-K-2}), 0\} S_{T-K-1} \\
&\quad + V_{T-K}(P_{T-K-1}, W_{T-K})] \\
&= \min_{\{S_{T-K-1}\}} E_{T-K-1} [\max \{(\theta S_{T-K-1} + \varepsilon_{T-K-1}), 0\} S_{T-K-1} \\
&\quad + V_{T-K}(P_{T-K-2} + \theta S_{T-K-1} + \varepsilon_{T-K-2}, W_{T-K-1} - S_{T-K-1})]
\end{aligned}$$

$$\begin{aligned}
&= \min_{\{S_{T-K-1}\}} \left[S_{T-K-1} \sigma_\varepsilon \left\{ \frac{\theta S_{T-K-1}}{\sigma_\varepsilon} + \frac{\phi \left(\frac{\theta S_{T-K-1}}{\sigma_\varepsilon} \right)}{\Phi \left(\frac{\theta S_{T-K-1}}{\sigma_\varepsilon} \right)} \right\} \right. \\
&\quad \left. + (W_{T-K-1} - S_{T-K-1}) \sigma_\varepsilon \left\{ \frac{\theta (W_{T-K-1} - S_{T-K-1})}{(K+1) \sigma_\varepsilon} + \frac{\phi \left[\frac{\theta (W_{T-K-1} - S_{T-K-1})}{(K+1) \sigma_\varepsilon} \right]}{\Phi \left[\frac{\theta (W_{T-K-1} - S_{T-K-1})}{(K+1) \sigma_\varepsilon} \right]} \right\} \right] \\
&= \min_{\{S_{T-K-1}\}} \left[S_{T-K-1} \left\{ \xi S_{T-K-1} + \frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} \right\} \right. \\
&\quad \left. + (W_{T-K-1} - S_{T-K-1}) \left\{ \frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} + \frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \right\} \right]
\end{aligned}$$

First Order Conditions (FOC) give,

$$\begin{aligned}
&\left\{ \xi S_{T-K-1} + \frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} \right\} \\
&+ S_{T-K-1} \left\{ \xi - \xi^2 S_{T-K-1} \frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} - \xi \left[\frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} \right]^2 \right\} \\
&- \left\{ \frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} + \frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \right\} \\
&+ (W_{T-K-1} - S_{T-K-1}) \left\{ -\frac{\xi}{(K+1)} + \frac{\xi^2 (W_{T-K-1} - S_{T-K-1})}{(K+1)^2} \frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \right. \\
&\quad \left. + \frac{\xi}{(K+1)} \left[\frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \right]^2 \right\} = 0
\end{aligned}$$

$$\begin{aligned}
&2\xi S_{T-K-1} + \frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} + \frac{\xi^2 (W_{T-K-1} - S_{T-K-1})^2}{(K+1)^2} \frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \\
&+ \frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \left[\frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]} \right]^2 = \\
&\xi^2 S_{T-K-1}^2 \frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} + \xi S_{T-K-1} \left[\frac{\phi (\xi S_{T-K-1})}{\Phi (\xi S_{T-K-1})} \right]^2 \\
&+ \frac{2\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} + \frac{\phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}{\Phi \left[\frac{\xi (W_{T-K-1} - S_{T-K-1})}{(K+1)} \right]}
\end{aligned}$$

This gives, $S_{T-K-1} = W_{T-K-1}/(K+2)$ and the corresponding value function is,

$$\begin{aligned}
V_{T-K-1}(P_{T-K-2}, W_{T-K-1}) &= \frac{W_{T-K-1}}{(K+2)} \sigma_\varepsilon \left\{ \frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)} \right\} \\
&+ \left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)} \right) \sigma_\varepsilon \left\{ \frac{\theta}{(K+1)\sigma_\varepsilon} \left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)} \right) \right. \\
&\left. + \frac{\phi\left[\frac{\theta}{(K+1)\sigma_\varepsilon} \left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)} \right)\right]}{\Phi\left[\frac{\theta}{(K+1)\sigma_\varepsilon} \left(W_{T-K-1} - \frac{W_{T-K-1}}{(K+2)} \right)\right]} \right\} \\
V_{T-K-1}(P_{T-K-2}, W_{T-K-1}) &= \sigma_\varepsilon W_{T-K-1} \left[\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)} + \frac{\phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)}{\Phi\left(\frac{\theta}{\sigma_\varepsilon} \frac{W_{T-K-1}}{(K+2)}\right)} \right]
\end{aligned}$$

This completes the induction. □

7.4 Proof of Proposition 4

Proof. Consider,

$$V_T(P_{T-1}, W_T) = E_T [\max\{(\theta W_T + \varepsilon_T), 0\} W_T]$$

$$V_T(P_{T-1}, W_T) = E_T [\max\{(\theta W_T^2 + W_T \varepsilon_T), 0\}]$$

$$V_T(P_{T-1}, W_T) = E_T [(\theta W_T^2 + W_T \varepsilon_T) | (\theta W_T^2 + W_T \varepsilon_T) > 0]$$

$$\{\cdot \mid E[\max(X, c)] = E[X | X > c] \Pr[X > c] + E[c | X \leq c] \Pr[X \leq c]\}$$

This is of the form, $E[Y | Y > 0]$ where, $Y = (\theta W_T^2 + W_T \varepsilon_T)$. We then need to calculate,

$$E_T \left[(\theta W_T^2 + W_T \sigma_\varepsilon Z) \mid Z > \left(-\frac{\theta W_T}{\sigma_\varepsilon} \right) \right], \text{ where } Z \sim N(0, 1)$$

$$[\cdot \mid Y \sim N(\theta W_T^2, W_T^2 \sigma_\varepsilon^2) \equiv Y \sim N(\mu, \sigma^2) \Rightarrow Y = \mu + \sigma Z; Y > 0 \Rightarrow Z > -\mu/\sigma]$$

We have for every standard normal distribution, Z , and for every u , $Pr[Z > -u] = Pr[Z < u] = \Phi(u)$. Here, ϕ and Φ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E[Z | Z > -u] &= \frac{1}{\Phi(u)} \left[\int_{-u}^{\infty} t \phi(t) dt \right] \\ &= \frac{1}{\Phi(u)} [-\phi(t)]_{-u}^{\infty} = \frac{\phi(u)}{\Phi(u)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E[Y | Y > 0] &= \mu + \sigma E \left[Z | Z > \left(-\frac{\mu}{\sigma} \right) \right] \\ &= \mu + \frac{\sigma \phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \end{aligned}$$

Setting, $\psi(u) = u + \phi(u)/\Phi(u)$,

$$E[Y | Y > 0] = \sigma \psi(\mu/\sigma)$$

$$\begin{aligned} V_T(P_{T-1}, W_T) &= E_T \left[(\theta W_T^2 + W_T \varepsilon_T) | (\theta W_T^2 + W_T \varepsilon_T) > 0 \right] \\ &= W_T \sigma_\varepsilon \left[\frac{\theta W_T}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta W_T}{\sigma_\varepsilon}\right)} \right] = \sigma_\varepsilon W_T \psi(\xi W_T), \quad \xi = \frac{\theta}{\sigma_\varepsilon} \end{aligned}$$

In the next to last period, $T-1$, the Bellman equation is,

$$\begin{aligned} V_{T-1}(P_{T-2}, W_{T-1}) &= \min_{\{S_{T-1}\}} E_{T-1} [\max\{(P_{T-1} - P_{T-2}), 0\} W_{T-1} + V_T(P_{T-1}, W_T)] \\ &= \min_{\{S_{T-1}\}} E_{T-1} [\max\{(\theta S_{T-1} + \varepsilon_{T-1}), 0\} W_{T-1} + V_T(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1}, W_{T-1} - S_{T-1})] \\ &= \min_{\{S_{T-1}\}} \left\{ W_{T-1} \sigma_\varepsilon \left[\frac{\theta S_{T-1}}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta S_{T-1}}{\sigma_\varepsilon}\right)} \right] + (W_{T-1} - S_{T-1}) \sigma_\varepsilon \left[\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon} + \frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1})}{\sigma_\varepsilon}\right)} \right] \right\} \\ &= \min_{\{S_{T-1}\}} [W_{T-1} \sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1}) \sigma_\varepsilon \psi(\xi (W_{T-1} - S_{T-1}))] \end{aligned}$$

We can show the above expression to be a convex function with a unique minimum since it is the sum of the portion shown to be convex earlier, a convex function and a linear component.

First Order Conditions (FOC) give,

$$\frac{\partial}{\partial S_{T-1}} [W_{T-1} \sigma_\varepsilon \psi(\xi S_{T-1}) + (W_{T-1} - S_{T-1}) \sigma_\varepsilon \psi(\xi (W_{T-1} - S_{T-1}))] = 0$$

$$\xi W_{T-1} \psi'(\xi S_{T-1}) - \xi (W_{T-1} - S_{T-1}) \psi'(\xi (W_{T-1} - S_{T-1})) - \psi(\xi (W_{T-1} - S_{T-1})) = 0$$

$$\begin{aligned}
& (\xi W_{T-1}) \left\{ 1 - \frac{\xi S_{T-1} \phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} - \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right]^2 \right\} \\
& - \xi (W_{T-1} - S_{T-1}) \left\{ 1 - \frac{\xi (W_{T-1} - S_{T-1}) \phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} - \left[\frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \right]^2 \right\} \\
& - \left\{ \xi \{W_{T-1} - S_{T-1}\} + \frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \right\} = 0
\end{aligned}$$

$$\left[\because \psi'(u) = 1 - \frac{u\phi(u)}{\Phi(u)} - \left[\frac{\phi(u)}{\Phi(u)} \right]^2 \text{ and } \psi(u) = u + \frac{\phi(u)}{\Phi(u)} \right]$$

$$\begin{aligned}
& W_{T-1} + \frac{\xi (W_{T-1} - S_{T-1})^2 \phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} + (W_{T-1} - S_{T-1}) \left[\frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} \right]^2 = \\
& (W_{T-1} - S_{T-1}) + \{W_{T-1} - S_{T-1}\} + \frac{1}{\xi} \frac{\phi(\xi \{W_{T-1} - S_{T-1}\})}{\Phi(\xi \{W_{T-1} - S_{T-1}\})} + \frac{\xi W_{T-1} S_{T-1} \phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} + W_{T-1} \left[\frac{\phi(\xi S_{T-1})}{\Phi(\xi S_{T-1})} \right]^2
\end{aligned}$$

□

7.5 Proof of Proposition 5

Proof. Consider,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T [\max \{(\theta W_T + \varepsilon_T + \gamma X_T), 0\} | W_T]$$

$$\begin{aligned}
V_T(P_{T-1}, X_{T-1}, W_T) &= E_T [\max \{(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T), 0\}] \\
&= E_T [(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) | (\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) > 0]
\end{aligned}$$

$$\{\because E[\max(X, c)] = E[X | X > c] Pr[X > c] + E[c | X \leq c] Pr[X \leq c]\}$$

This is of the form, $E[Y | Y > 0]$ where, $Y = (\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T)$. We then need to calculate,

$$E_T \left[\left\{ \theta W_T^2 + \gamma \rho W_T X_{T-1} + W_T \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) Z \right\} \middle| Z > \left(-\frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right], \text{ where } Z \sim N(0, 1)$$

$$[\because X \sim N(\mu_X, \sigma_X^2); Y \sim N(\mu_Y, \sigma_Y^2); U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)]$$

$$[\because Y \sim N(\theta W_T^2 + \gamma \rho W_T X_{T-1}, W_T^2 \{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2\}) \equiv Y \sim N(\mu, \sigma^2) \Rightarrow Y = \mu + \sigma Z; Y > 0 \Rightarrow Z > -\mu/\sigma]$$

We have for every standard normal distribution, Z , and for every u , $Pr[Z > -u] = Pr[Z < u] = \Phi(u)$.

Here, ϕ and Φ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E[Z|Z > -u] &= \frac{1}{\Phi(u)} \left[\int_{-u}^{\infty} t\phi(t) dt \right] \\ &= \frac{1}{\Phi(u)} [-\phi(t)|_{-u}^{\infty}] = \frac{\phi(u)}{\Phi(u)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E[Y|Y > 0] &= \mu + \sigma E\left[Z|Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma\phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \end{aligned}$$

Setting, $\psi(u) = u + \phi(u)/\Phi(u)$,

$$E[Y|Y > 0] = \sigma\psi(\mu/\sigma)$$

$$\begin{aligned} V_T(P_{T-1}, X_{T-1}, W_T) &= E_T[(\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T \eta_T) | (\theta W_T^2 + W_T\varepsilon_T + \gamma\rho W_T X_{T-1} + \gamma W_T \eta_T) > 0] \\ &= W_T \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)} \right] \\ &= \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) W_T \psi(\xi W_T), \quad \xi W_T = \frac{\theta W_T + \gamma\rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \end{aligned}$$

In the next to last period, $T-1$, the Bellman equation is,

$$\begin{aligned} V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1}) &= \min_{\{S_{T-1}\}} E_{T-1}[\max\{(P_{T-1} - P_{T-2}), 0\} S_{T-1} + V_T(P_{T-1}, X_{T-1}, W_T)] \\ &= \min_{\{S_{T-1}\}} E_{T-1}[\max\{(\theta S_{T-1}^2 + S_{T-1}\varepsilon_{T-1} + \gamma\rho S_{T-1} X_{T-2} + \gamma S_{T-1} \eta_{T-1}), 0\} \\ &\quad + V_T(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1} + \gamma\rho X_{T-2} + \gamma \eta_{T-1}, \rho X_{T-2} + \eta_{T-1}, W_{T-1} - S_{T-1})] \\ &= \min_{\{S_{T-1}\}} \left\{ S_{T-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta S_{T-1} + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)} \right] \right. \\ &\quad \left. + \{W_{T-1} - S_{T-1}\} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta(W_{T-1} - S_{T-1}) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta(W_{T-1} - S_{T-1}) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta(W_{T-1} - S_{T-1}) + \gamma\rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)} \right] \right\} \end{aligned}$$

$$= \min_{\{S_{T-1}\}} \left[S_{T-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi(\xi_1 S_{T-1}) + (W_{T-1} - S_{T-1}) \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi\{\xi_1 (W_{T-1} - S_{T-1})\} \right]$$

$$\text{Here, } \xi_1 S_{T-1} = \frac{\theta S_{T-1} + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \text{ Also, let } \alpha_1 = \gamma \rho X_{T-2}, \beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$$

$$= \min_{\{S_{T-1}\}} \left\{ \left[\theta S_{T-1}^2 + \alpha_1 S_{T-1} + \beta S_{T-1} \frac{\phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)} \right] \right. \\ \left. + \left[\theta (W_{T-1} - S_{T-1})^2 + \alpha_1 (W_{T-1} - S_{T-1}) + \beta (W_{T-1} - S_{T-1}) \frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)} \right] \right\}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking First Order Conditions give,

$$2\theta S_{T-1} + \alpha_1 + \beta \left\{ \frac{\phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)} + \frac{\theta S_{T-1}}{\beta} \left[-\frac{\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right) \phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)} - \left\{ \frac{\phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-1} + \alpha_1}{\beta}\right)} \right\}^2 \right] \right\} \\ - 2\theta (W_{T-1} - S_{T-1}) - \alpha_1 + \beta \left\{ -\frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)} \right. \\ \left. + \frac{\theta (W_{T-1} - S_{T-1})}{\beta} \left[\frac{\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right) \phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)} + \left\{ \frac{\phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)}{\Phi\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha_1}{\beta}\right)} \right\}^2 \right] \right\} = 0$$

$S_{T-1} = W_{T-1}/2$ solves this, giving the value function,

$$V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1}) = \frac{\theta}{2} W_{T-1}^2 + \alpha_1 W_{T-1} + \beta W_{T-1} \frac{\phi\left(\frac{\theta W_{T-1} + 2\alpha_1}{2\beta}\right)}{\Phi\left(\frac{\theta W_{T-1} + 2\alpha_1}{2\beta}\right)}$$

Continuing the recursion,

$$V_{T-2}(P_{T-3}, X_{T-3}, W_{T-2}) = \min_{\{S_{T-2}\}} E_{T-2} [\max\{(P_{T-2} - P_{T-3}), 0\} S_{T-2} + V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1})] \\ = \min_{\{S_{T-2}\}} E_{T-2} [\max\{(\theta S_{T-2}^2 + S_{T-2} \varepsilon_{T-2} + \gamma \rho S_{T-2} X_{T-3} + \gamma S_{T-2} \eta_{T-2}), 0\} \\ + V_{T-1}(P_{T-3} + \theta S_{T-2} + \varepsilon_{T-2} + \gamma \rho X_{T-3} + \gamma \eta_{T-2}, \rho X_{T-3} + \eta_{T-2}, W_{T-2} - S_{T-2})]$$

$$\begin{aligned}
&= \min_{\{S_{T-2}\}} \left\{ S_{T-2} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta S_{T-2} + \gamma \rho X_{T-3}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta S_{T-2} + \gamma \rho X_{T-3}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta S_{T-2} + \gamma \rho X_{T-3}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right. \\
&\quad \left. + \{W_{T-2} - S_{T-2}\} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta \{W_{T-2} - S_{T-2}\} + 2\gamma \rho X_{T-3}}{2\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta \{W_{T-2} - S_{T-2}\} + 2\gamma \rho X_{T-3}}{2\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta \{W_{T-2} - S_{T-2}\} + 2\gamma \rho X_{T-3}}{2\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right\} \\
&= \min_{\{S_{T-2}\}} \left[S_{T-2} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi(\xi_2 S_{T-2}) + (W_{T-2} - S_{T-2}) \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi\{\xi_2 (W_{T-2} - S_{T-2})\} \right]
\end{aligned}$$

$$\text{Here, } \xi_2 S_{T-2} = \frac{\theta S_{T-2} + 2\gamma \rho X_{T-3}}{2\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \text{ Also, let } \alpha_2 = \gamma \rho X_{T-3}, \beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$$

$$\begin{aligned}
&= \min_{\{S_{T-2}\}} \left\{ \left[\theta S_{T-2}^2 + \alpha_2 S_{T-2} + \beta S_{T-2} \frac{\phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)} \right] \right. \\
&\quad \left. + \left[\frac{\theta}{2} (W_{T-2} - S_{T-2})^2 + \alpha_2 (W_{T-2} - S_{T-2}) + \beta (W_{T-2} - S_{T-2}) \frac{\phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)}{\Phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)} \right] \right\}
\end{aligned}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking First Order Conditions give,

$$\begin{aligned}
&2\theta S_{T-2} + \alpha_2 + \beta \left\{ \frac{\phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)} + \frac{\theta S_{T-2}}{\beta} \left[-\frac{\left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right) \phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)} - \left\{ \frac{\phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-2} + \alpha_2}{\beta} \right)} \right\}^2 \right] \right\} \\
&\quad - \theta (W_{T-2} - S_{T-2}) - \alpha_2 + \beta \left\{ -\frac{\phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)}{\Phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)} \right. \\
&\quad \left. + \frac{\theta (W_{T-2} - S_{T-2})}{2\beta} \left[\frac{\left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right) \phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)}{\Phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)} + \left\{ \frac{\phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)}{\Phi \left(\frac{\theta (W_{T-2} - S_{T-2}) + 2\alpha_2}{2\beta} \right)} \right\}^2 \right] \right\} = 0
\end{aligned}$$

$S_{T-1} = W_{T-1}/3$ solves this, giving the value function,

$$V_{T-2}(P_{T-3}, X_{T-3}, W_{T-2}) = \frac{\theta}{3} W_{T-2}^2 + \alpha_2 W_{T-2} + \beta W_{T-2} \frac{\phi \left(\frac{\theta W_{T-2} + 3\alpha_2}{3\beta} \right)}{\Phi \left(\frac{\theta W_{T-2} + 3\alpha_2}{3\beta} \right)}$$

We show the general case using induction. Let the value function hold for $T - K$

$$V_{T-K}(P_{T-K-1}, X_{T-K-1}, W_{T-K}) = \frac{\theta}{(K+1)} W_{T-K}^2 + \alpha_K W_{T-K} + \beta W_{T-K} \frac{\phi \left(\frac{\theta W_{T-K} + (K+1)\alpha_K}{(K+1)\beta} \right)}{\Phi \left(\frac{\theta W_{T-K} + (K+1)\alpha_K}{(K+1)\beta} \right)}$$

$$\begin{aligned}
V_{T-K-1}(P_{T-K-2}, X_{T-K-2}, W_{T-K-1}) &= \min_{\{S_{T-K-1}\}} E_{T-K-1} [\max \{(P_{T-K-1} - P_{T-K-2}), 0\} S_{T-K-1} \\
&\quad + V_{T-K}(P_{T-K-1}, X_{T-K-1}, W_{T-K})] \\
&= \min_{\{S_{T-K-1}\}} E_{T-K-1} [\max \{(\theta S_{T-K-1}^2 + S_{T-K-1} \varepsilon_{T-K-1} + \gamma \rho S_{T-K-1} X_{T-K-2} + \gamma S_{T-K-1} \eta_{T-K-1}), 0\} \\
&\quad + V_{T-K}(P_{T-K-2} + \theta S_{T-K-1} + \varepsilon_{T-K-1} + \gamma \rho X_{T-K-2} + \gamma \eta_{T-K-1}, \rho X_{T-K-2} + \eta_{T-K-1}, W_{T-K-1} - S_{T-K-1})]
\end{aligned}$$

$$\begin{aligned}
&= \min_{\{S_{T-K-1}\}} \left\{ S_{T-K-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta S_{T-K-1} + \gamma \rho X_{T-K-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta S_{T-K-1} + \gamma \rho X_{T-K-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta S_{T-K-1} + \gamma \rho X_{T-K-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right. \\
&\quad \left. + \{W_{T-K-1} - S_{T-K-1}\} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \right. \\
&\quad \left. \left[\frac{\theta \{W_{T-K-1} - S_{T-K-1}\} + (K+1) \gamma \rho X_{T-K-2}}{(K+1) \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta \{W_{T-K-1} - S_{T-K-1}\} + (K+1) \gamma \rho X_{T-K-2}}{(K+1) \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta \{W_{T-K-1} - S_{T-K-1}\} + (K+1) \gamma \rho X_{T-K-2}}{(K+1) \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right\} \\
&= \min_{\{S_{T-K-1}\}} \left[S_{T-K-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi(\xi_{K+1} S_{T-K-1}) \right. \\
&\quad \left. + (W_{T-K-1} - S_{T-K-1}) \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi\{\xi_{K+1} (W_{T-K-1} - S_{T-K-1})\} \right]
\end{aligned}$$

Here, $\xi_{K+1} S_{T-K-1} = \frac{\theta S_{T-K-1} + 2\gamma \rho X_{T-K-2}}{2\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}$ Also, let $\alpha_{K+1} = \gamma \rho X_{T-K-2}$, $\beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$

$$\begin{aligned}
&= \min_{\{S_{T-K-1}\}} \left\{ \left[\theta S_{T-K-1}^2 + \alpha_{K+1} S_{T-K-1} + \beta S_{T-K-1} \frac{\phi \left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta} \right)} \right] \right. \\
&\quad \left. + \left[\frac{\theta}{(K+1)} (W_{T-K-1} - S_{T-K-1})^2 + \alpha_{K+1} (W_{T-K-1} - S_{T-K-1}) \right] \right. \\
&\quad \left. + \left[\beta (W_{T-K-1} - S_{T-K-1}) \frac{\phi \left(\frac{\theta (W_{T-K-1} - S_{T-K-1}) + (K+1) \alpha_{K+1}}{(K+1) \beta} \right)}{\Phi \left(\frac{\theta (W_{T-K-1} - S_{T-K-1}) + (K+1) \alpha_{K+1}}{(K+1) \beta} \right)} \right] \right\}
\end{aligned}$$

This is a convex function and numerical solutions can be obtained at each stage of the recursion or taking

First Order Conditions give,

$$\begin{aligned}
& 2\theta S_{T-K-1} + \alpha_{K+1} + \beta \left\{ \frac{\phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)} \right. \\
& + \frac{\theta S_{T-K-1}}{\beta} \left[-\frac{\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right) \phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)} - \left\{ \frac{\phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)}{\Phi\left(\frac{\theta S_{T-K-1} + \alpha_{K+1}}{\beta}\right)} \right\}^2 \right] \Bigg\} \\
& - \frac{2\theta}{(K+1)} (W_{T-K-1} - S_{T-K-1}) - \alpha_{K+1} + \beta \left\{ -\frac{\phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right. \\
& + \frac{\theta(W_{T-K-1} - S_{T-K-1})}{(K+1)\beta} \left[\frac{\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right) \phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right. \\
& \left. \left. + \left\{ \frac{\phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)}{\Phi\left(\frac{\theta(W_{T-K-1} - S_{T-K-1}) + (K+1)\alpha_{K+1}}{(K+1)\beta}\right)} \right\}^2 \right] \right\} = 0
\end{aligned}$$

$S_{T-K-1} = W_{T-K-1} / (K+2)$ solves this, giving the value function and completing the induction.

$$V_{T-K-1}(P_{T-K-2}, X_{T-K-2}, W_{T-K-1}) = \frac{\theta}{(K+2)} W_{T-K-1}^2 + \alpha_{K+1} W_{T-K-1} + \beta W_{T-K-1} \frac{\phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}{\Phi\left(\frac{\theta W_{T-K-1} + (K+2)\alpha_{K+1}}{(K+2)\beta}\right)}$$

□

7.6 Proof of Proposition 6

Proof. Consider,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T[\max\{(\theta W_T + \varepsilon_T + \gamma X_T), 0\} | W_T]$$

$$\begin{aligned}
V_T(P_{T-1}, X_{T-1}, W_T) &= E_T[\max\{(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T), 0\}] \\
&= E_T[(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) | (\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) > 0] \\
&= \{ \cdot \cdot E[\max(X, c)] = E[X | X > c] Pr[X > c] + E[c | X \leq c] Pr[X \leq c] \}
\end{aligned}$$

This is of the form, $E[Y | Y > 0]$ where, $Y = (\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T)$. We then need to calculate,

$$E_T \left[\left\{ \theta W_T^2 + \gamma \rho W_T X_{T-1} + W_T \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) Z \right\} \middle| Z > \left(-\frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right], \text{ where } Z \sim N(0, 1)$$

$$[\because X \sim N(\mu_X, \sigma_X^2); Y \sim N(\mu_Y, \sigma_Y^2); U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)]$$

$$[\because Y \sim N(\theta W_T^2 + \gamma \rho W_T X_{T-1}, W_T^2 \{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2\}) \equiv Y \sim N(\mu, \sigma^2) \Rightarrow Y = \mu + \sigma Z; Y > 0 \Rightarrow Z > -\mu/\sigma]$$

We have for every standard normal distribution, Z , and for every u , $Pr[Z > -u] = Pr[Z < u] = \Phi(u)$.

Here, ϕ and Φ are the standard normal PDF and CDF, respectively.

$$\begin{aligned} E[Z | Z > -u] &= \frac{1}{\Phi(u)} \left[\int_{-u}^{\infty} t \phi(t) dt \right] \\ &= \frac{1}{\Phi(u)} [-\phi(t)|_{-u}^{\infty}] = \frac{\phi(u)}{\Phi(u)} \end{aligned}$$

Hence we have,

$$\begin{aligned} E[Y | Y > 0] &= \mu + \sigma E\left[Z | Z > \left(-\frac{\mu}{\sigma}\right)\right] \\ &= \mu + \frac{\sigma \phi(\mu/\sigma)}{\Phi(\mu/\sigma)} \end{aligned}$$

Setting, $\psi(u) = u + \phi(u)/\Phi(u)$,

$$E[Y | Y > 0] = \sigma \psi(\mu/\sigma)$$

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T[(\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) | (\theta W_T^2 + W_T \varepsilon_T + \gamma \rho W_T X_{T-1} + \gamma W_T \eta_T) > 0]$$

$$\begin{aligned} &= W_T \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi\left(\frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)}{\Phi\left(\frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}\right)} \right] \\ &= \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) W_T \psi(\xi W_T), \quad \xi W_T = \frac{\theta W_T + \gamma \rho X_{T-1}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \end{aligned}$$

In the next to last period, $T-1$, the Bellman equation is,

$$V_{T-1}(P_{T-2}, X_{T-2}, W_{T-1}) = \min_{\{S_{T-1}\}} E_{T-1}[\max\{(P_{T-1} - P_{T-2}), 0\} W_{T-1} + V_T(P_{T-1}, X_{T-1}, W_T)]$$

$$\begin{aligned} &= \min_{\{S_{T-1}\}} E_{T-1}[\max\{(\theta W_{T-1} S_{T-1} + W_{T-1} \varepsilon_{T-1} + \gamma \rho W_{T-1} X_{T-2} + \gamma W_{T-1} \eta_{T-1}), 0\} \\ &\quad + V_T(P_{T-2} + \theta S_{T-1} + \varepsilon_{T-1} + \gamma \rho X_{T-2} + \gamma \eta_{T-1}, \rho X_{T-2} + \eta_{T-1}, W_{T-1} - S_{T-1})] \end{aligned}$$

$$\begin{aligned}
&= \min_{\{S_{T-1}\}} \left\{ W_{T-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta S_{T-1} + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta S_{T-1} + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta S_{T-1} + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right. \\
&\quad \left. + \{W_{T-1} - S_{T-1}\} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\frac{\theta (W_{T-1} - S_{T-1}) + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} + \frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \right\} \\
&= \min_{\{S_{T-1}\}} \left[W_{T-1} \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi(\xi_1 S_{T-1}) + (W_{T-1} - S_{T-1}) \left(\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \psi\{\xi_1 (W_{T-1} - S_{T-1})\} \right]
\end{aligned}$$

Here, $\xi_1 S_{T-1} = \frac{\theta S_{T-1} + \gamma \rho X_{T-2}}{\sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}}$ Also, let $\alpha = \gamma \rho X_{T-2}$, $\beta = \sqrt{\gamma^2 \sigma_\eta^2 + \sigma_\varepsilon^2}$

$$\begin{aligned}
&= \min_{\{S_{T-1}\}} \left\{ \left[\theta W_{T-1} S_{T-1} + \alpha W_{T-1} + \beta W_{T-1} \frac{\phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)} \right] \right. \\
&\quad \left. + \left[\theta (W_{T-1} - S_{T-1})^2 + \alpha (W_{T-1} - S_{T-1}) + \beta (W_{T-1} - S_{T-1}) \frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} \right] \right\}
\end{aligned}$$

This is a convex function and taking First Order Conditions give,

$$\begin{aligned}
&\theta W_{T-1} + \beta W_{T-1} \left\{ \frac{\theta}{\beta} \left[-\frac{\left(\frac{\theta S_{T-1} + \alpha}{\beta} \right) \phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)} - \left\{ \frac{\phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta S_{T-1} + \alpha}{\beta} \right)} \right\}^2 \right] \right\} \\
&\quad - 2\theta (W_{T-1} - S_{T-1}) - \alpha + \beta \left\{ -\frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} \right. \\
&\quad \left. + \frac{\theta (W_{T-1} - S_{T-1})}{\beta} \left[\frac{\left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right) \phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} + \left\{ \frac{\phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)}{\Phi \left(\frac{\theta (W_{T-1} - S_{T-1}) + \alpha}{\beta} \right)} \right\}^2 \right] \right\} = 0
\end{aligned}$$

□

7.7 Proof of Proposition 7

Proof. Consider,

$$V_T(P_{T-1}, X_{T-1}, W_T) = E_T \left[\max \left\{ \left(\tilde{P}_T (1 + \theta W_T + \gamma X_T) - P_{T-1} \right), 0 \right\} W_T \right]$$

$$\begin{aligned}
V_T(P_{T-1}, X_{T-1}, W_T) &= E_T \left[\max \left\{ \left(\tilde{P}_{T-1} e^{B_T} [1 + \theta W_T + \gamma (\rho X_{T-1} + \eta_T)] - P_{T-1} \right), 0 \right\} W_T \right] \\
&= E_T \left[\max \left\{ \tilde{P}_{T-1} W_T e^{B_T} + \theta W_T^2 \tilde{P}_{T-1} e^{B_T} + \gamma \rho X_{T-1} \tilde{P}_{T-1} W_T e^{B_T} + \gamma \tilde{P}_{T-1} W_T e^{B_T} \eta_T - W_T P_{T-1}, 0 \right\} \right] \\
&= E_T \left[\left(\tilde{P}_{T-1} W_T e^{B_T} + \theta W_T^2 \tilde{P}_{T-1} e^{B_T} + \gamma \rho X_{T-1} \tilde{P}_{T-1} W_T e^{B_T} + \gamma \tilde{P}_{T-1} W_T e^{B_T} \eta_T - W_T P_{T-1} \right) \right] \\
&\quad \left(\tilde{P}_{T-1} W_T e^{B_T} + \theta W_T^2 \tilde{P}_{T-1} e^{B_T} + \gamma \rho X_{T-1} \tilde{P}_{T-1} W_T e^{B_T} + \gamma \tilde{P}_{T-1} W_T e^{B_T} \eta_T - W_T P_{T-1} \right) > 0 \Big]
\end{aligned}$$

$$\{\cdot \mid E[\max(X, c)] = E[X \mid X > c] \Pr[X > c] + E[c \mid X \leq c] \Pr[X \leq c]\}$$

This is of the form, $E[Y_2 \mid Y_2 > 0]$ where,

$Y_2 = \left(\tilde{P}_{T-1} W_T e^{B_T} + \theta W_T^2 \tilde{P}_{T-1} e^{B_T} + \gamma \rho X_{T-1} \tilde{P}_{T-1} W_T e^{B_T} + \gamma \tilde{P}_{T-1} W_T e^{B_T} \eta_T - W_T P_{T-1} \right)$. We simplify using some notational shortcuts,

$$E \left[(ae^X + be^X + ce^X + de^X Y_1 + k) \mid (ae^X + be^X + ce^X + de^X Y_1 + k) > 0 \right]$$

$X \sim N(\mu_X, \sigma_X^2); Y_1 \sim N(0, \sigma_{Y_1}^2); X$ and Y_1 are independent. Also, $a, b, c, d > 0, k < 0$

$$\equiv E \left[(e^X \{a + b + c + dY_1\} + k) \mid (e^X \{a + b + c + dY_1\} + k) > 0 \right]$$

$$\equiv E \left[(e^X Y + k) \mid (e^X Y + k) > 0 \right]$$

$X \sim N(\mu_X, \sigma_X^2); Y \sim N(\mu_Y, \sigma_Y^2); X$ and Y are independent. Also, $k < 0$

Consider,

$$E \left[(e^X Y + k) \mid (e^X Y + k) > 0 \right] = E \left[k \mid (e^X Y + k) > 0 \right] + E \left[(e^X Y) \mid (e^X Y + k) > 0 \right]$$

$$= k + E \left[(Y e^X) \mid (Y e^X + k) > 0 \right]$$

$$= k + \int \int y e^x f(y e^x \mid \{y e^x + k\} > 0) dx dy$$

Here, $f(w)$ is the probability density function for w ,

$$\begin{aligned}
&= k + \int \int y e^x \frac{f(y e^x; \{y e^x + k\} > 0)}{f(\{y e^x + k\} > 0)} dx dy \\
&\quad [\text{We note that, } y e^x > -k > 0 \Rightarrow y > 0] \\
&= k + \int \int y e^x \frac{f(y) f(e^x; \{y e^x + k\} > 0)}{f(\{y e^x + k\} > 0)} dx dy \\
&= k + \int y \left[\int \frac{e^x f(e^x; \{e^x > -\frac{k}{y}\})}{f(e^x > -\frac{k}{y})} dx \right] f(y) dy \\
&= k + \int y \left[\int e^x f\left(e^x \mid \left\{e^x > -\frac{k}{y}\right\}\right) dx \right] f(y) dy \\
&= k + \int_0^{(y < -k)} y \left[\int e^x f(e^x \mid \{e^x > 1\}) dx \right] f(y) dy + \int_{(y > -k)}^\infty y \left[\int e^x f(e^x \mid \{e^x < 1\}) dx \right] f(y) dy \\
&= k + \int_0^{(-k)} y [E(W \mid W > c)] f(y) dy + \int_{(-k)}^\infty y [E(W \mid W < c)] f(y) dy \quad ; \text{ here, } W = e^X \text{ and } c = 1
\end{aligned}$$

Simplifying the inner expectations,

$$E(W \mid W > c) = \frac{1}{P(e^X > c)} \int_c^\infty w \frac{1}{w \sigma_X \sqrt{2\pi}} e^{-\frac{1}{2} \left[\frac{\ln(w) - \mu_X}{\sigma_X} \right]^2} dw$$

Put $t = \ln(w)$, we have, $dw = e^t dt$

$$\begin{aligned}
E(W \mid W > c) &= \frac{1}{P(X > \ln(c))} \int_{\ln(c)}^\infty \frac{e^t}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2} \left(\frac{t - \mu_X}{\sigma_X} \right)^2} dt \\
t - \frac{1}{2} \left(\frac{t - \mu_X}{\sigma_X} \right)^2 &= -\frac{1}{2\sigma_X^2} (t - (\mu_X + \sigma_X^2))^2 + \mu_X + \frac{\sigma_X^2}{2} \\
E(W \mid W > c) &= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P(\mu_X + \sigma_X Z > \ln(c))} \int_{\ln(c)}^\infty \frac{1}{\sigma_X \sqrt{2\pi}} e^{-\frac{1}{2} \left[\frac{t - (\mu_X + \sigma_X^2)}{\sigma_X} \right]^2} dt \quad ; Z \sim N(0, 1)
\end{aligned}$$

Put $s = \left\lceil \frac{t - (\mu_X + \sigma_X^2)}{\sigma_X} \right\rceil$ and $b = \left\lceil \frac{\ln(c) - (\mu_X + \sigma_X^2)}{\sigma_X} \right\rceil$ we have, $ds = \frac{dt}{\sigma_X}$

$$\begin{aligned}
E(W|W > c) &= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P\left(Z > \frac{\ln(c) - \mu_X}{\sigma_X}\right)} \int_b^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds \\
&= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P\left(Z < \frac{-\ln(c) + \mu_X}{\sigma_X}\right)} \left[\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds - \int_{-\infty}^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds \right] \\
&= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P\left(Z < \frac{-\ln(c) + \mu_X}{\sigma_X}\right)} [1 - \Phi(b)] \quad ; \Phi \text{ is the standard normal CDF} \\
&= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{\Phi\left(\frac{-\ln(c) + \mu_X}{\sigma_X}\right)} [\Phi(-b)]
\end{aligned}$$

Similarly for the other case,

$$E(W|W < c) = \frac{1}{P(e^X < c)} \int_0^c w \frac{1}{w\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{\ln(w) - \mu_X}{\sigma_X}\right]^2} dw$$

Put $t = \ln(w)$, we have, $dw = e^t dt$

$$\begin{aligned}
E(W|W < c) &= \frac{1}{P(X < \ln(c))} \int_{-\infty}^{\ln(c)} \frac{e^t}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t - \mu_X}{\sigma_X}\right)^2} dt \\
t - \frac{1}{2}\left(\frac{t - \mu_X}{\sigma_X}\right)^2 &= -\frac{1}{2\sigma_X^2} (t - (\mu_X + \sigma_X^2))^2 + \mu_X + \frac{\sigma_X^2}{2} \\
E(W|W < c) &= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P(\mu_X + \sigma_X Z < \ln(c))} \int_{-\infty}^{\ln(c)} \frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{t - (\mu_X + \sigma_X^2)}{\sigma_X}\right]^2} dt \quad ; Z \sim N(0, 1)
\end{aligned}$$

Put $s = \left\lceil \frac{t - (\mu_X + \sigma_X^2)}{\sigma_X} \right\rceil$ and $b = \left\lceil \frac{\ln(c) - (\mu_X + \sigma_X^2)}{\sigma_X} \right\rceil$ we have, $ds = \frac{dt}{\sigma_X}$

$$\begin{aligned}
E(W|W < c) &= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P\left(Z < \frac{\ln(c) - \mu_X}{\sigma_X}\right)} \int_{-\infty}^b \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2} ds \\
&= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{P\left(Z < \frac{\ln(c) - \mu_X}{\sigma_X}\right)} [\Phi(b)] \quad ; \Phi \text{ is the standard normal CDF} \\
&= \frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{\Phi\left(\frac{\ln(c) - \mu_X}{\sigma_X}\right)} [\Phi(b)]
\end{aligned}$$

Using the results for the inner expectations,

$$\begin{aligned}
E[(e^X Y + k) | (e^X Y + k) > 0] &= k + \int_0^{(-k)} y \left[\frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{\Phi\left(\frac{-\ln(c) + \mu_X}{\sigma_X}\right)} [\Phi(-b)] \right] f(y) dy + \int_{(-k)}^\infty y \left[\frac{e^{(\mu_X + \frac{1}{2}\sigma_X^2)}}{\Phi\left(\frac{\ln(c) - \mu_X}{\sigma_X}\right)} [\Phi(b)] \right] f(y) dy \\
&= k + e^{(\mu_X + \frac{1}{2}\sigma_X^2)} \left[\int_0^{(-k)} y \left\{ \frac{\Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} f(y) dy + \int_{(-k)}^\infty y \left\{ \frac{\Phi\left(-\left[\frac{\mu_X + \sigma_X^2}{\sigma_X}\right]\right)}{\Phi\left(-\left[\frac{\mu_X}{\sigma_X}\right]\right)} \right\} f(y) dy \right] \\
&= k + e^{(\mu_X + \frac{1}{2}\sigma_X^2)} \left[\int_0^{(-k)} y \left\{ \frac{\Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} f(y) dy + \int_{(-k)}^\infty y \left\{ \frac{1 - \Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{1 - \Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} f(y) dy \right] \\
&= k + e^{(\mu_X + \frac{1}{2}\sigma_X^2)} \left[\left\{ \frac{\Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \int_{-\frac{\mu_Y}{\sigma_Y}}^{-(\frac{k + \mu_Y}{\sigma_Y})} (\mu_Y + \sigma_Y z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right. \\
&\quad \left. + \left\{ \frac{1 - \Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{1 - \Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \int_{-(\frac{k + \mu_Y}{\sigma_Y})}^\infty (\mu_Y + \sigma_Y z) \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} dz \right] ; Z \sim N(0, 1) \\
&= k + e^{(\mu_X + \frac{1}{2}\sigma_X^2)} \left[\left\{ \frac{\Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{\Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \left\{ \mu_Y \left[\Phi\left(-\left[\frac{k + \mu_Y}{\sigma_Y}\right]\right) - \Phi\left(-\frac{\mu_Y}{\sigma_Y}\right) \right] - \frac{\sigma_Y}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{k + \mu_Y}{\sigma_Y}\right)^2} - e^{-\frac{1}{2}\left(\frac{\mu_Y}{\sigma_Y}\right)^2} \right] \right\} \right. \\
&\quad \left. + \left\{ \frac{1 - \Phi\left(\frac{\mu_X + \sigma_X^2}{\sigma_X}\right)}{1 - \Phi\left(\frac{\mu_X}{\sigma_X}\right)} \right\} \left\{ \mu_Y \left[1 - \Phi\left(-\left[\frac{k + \mu_Y}{\sigma_Y}\right]\right) \right] + \frac{\sigma_Y}{\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{k + \mu_Y}{\sigma_Y}\right)^2} \right] \right\} \right]
\end{aligned}$$

□

7.8 Proof of Proposition 8

Proof. Consider,

$$V_T(P_{T-1}, O_{T-1}, W_T) = \min_{\{S_T\}} E_T[\max\{(P_T - P_{T-1}), 0\} S_T]$$

$$V_T(P_{T-1}, O_{T-1}, W_T) = E_T[\max\{(\alpha P_{T-1} + \theta W_T P_{T-1} - \gamma(O_T - W_T) P_{T-1} + \varepsilon_T), 0\} W_T]$$

$$= E_T[\max\{(\alpha P_{T-1} W_T + \theta W_T^2 P_{T-1} + \gamma W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T), 0\}]$$

Setting $\beta = \theta + \gamma$,

$$V_T(P_{T-1}, O_{T-1}, W_T) = E_T \left[(\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T) \mid \right. \\ \left. (\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T) > 0 \right]$$

$$\{ \cdot \mid E[\max(X, c)] = E[X \mid X > c] \Pr[X > c] + E[c \mid X \leq c] \Pr[X \leq c] \}$$

This is of the form, $E[Y \mid Y > 0]$ where,

$Y = (\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T)$. We then need to calculate,

$$E_T \left[\left\{ \alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} + W_T \left(\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) Z \right\} \mid \right. \\ \left. Z > \left(-\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right], \text{ where } Z \sim N(0, 1)$$

$$[\cdot \mid X \sim N(\mu_X, \sigma_X^2); Y \sim N(\mu_Y, \sigma_Y^2); U = X + Y \Rightarrow U \sim N(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)]$$

$$\left[\cdot \mid Y \sim N(\alpha P_{T-1} W_T + \beta W_T^2 P_{T-1} - \gamma \rho O_{T-1} W_T P_{T-1} - \gamma W_T P_{T-1} \eta_T + W_T \varepsilon_T) \right. \\ \left. \equiv Y \sim N(\mu, \sigma^2) \Rightarrow Y = \mu + \sigma Z; Y > 0 \Rightarrow Z > -\mu/\sigma \right]$$

We have for every standard normal distribution, Z , and for every u , $\Pr[Z > -u] = \Pr[Z < u] = \Phi(u)$.

Here, ϕ and Φ are the standard normal PDF and CDF, respectively.

$$E[Z \mid Z > -u] = \frac{1}{\Phi(u)} \left[\int_{-u}^{\infty} t \phi(t) dt \right] \\ = \frac{1}{\Phi(u)} [-\phi(t)]_{-u}^{\infty} = \frac{\phi(u)}{\Phi(u)}$$

Hence we have,

$$E[Y \mid Y > 0] = \mu + \sigma E \left[Z \mid Z > \left(-\frac{\mu}{\sigma} \right) \right] \\ = \mu + \frac{\sigma \phi(\mu/\sigma)}{\Phi(\mu/\sigma)}$$

Setting, $\psi(u) = u + \phi(u)/\Phi(u)$,

$$E[Y \mid Y > 0] = \sigma \psi(\mu/\sigma)$$

$$\begin{aligned}
V_T(P_{T-1}, O_{T-1}, W_T) &= W_T \left(\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \left[\left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right. \\
&\quad \left. + \frac{\phi \left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \\
&= \left(\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) W_T \psi(\xi W_T), \quad \xi W_T = \left(\frac{\alpha P_{T-1} + \beta W_T P_{T-1} - \gamma \rho O_{T-1} P_{T-1}}{\sqrt{\gamma^2 P_{T-1}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)
\end{aligned}$$

In the next to last period, $T-1$, the Bellman equation is,

$$\begin{aligned}
V_{T-1}(P_{T-2}, O_{T-2}, W_{T-1}) &= \min_{\{S_{T-1}\}} E_{T-1} [\max\{(P_{T-1} - P_{T-2}), 0\} S_{T-1} + V_T(P_{T-1}, O_{T-1}, W_T)] \\
&= \min_{\{S_{T-1}\}} E_{T-1} [\max\{(\alpha P_{T-2} S_{T-1} + \beta S_{T-1}^2 P_{T-2} - \gamma \rho O_{T-2} S_{T-1} P_{T-2} - \gamma S_{T-1} P_{T-2} \eta_{T-1} + S_{T-1} \varepsilon_{T-1}), 0\} \\
&\quad + V_T((\alpha + 1) P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2} - \gamma P_{T-2} \eta_{T-1} + \varepsilon_{T-1}, \rho O_{T-2} + \eta_{T-1}, W_{T-1} - S_{T-1})] \\
&= \min_{\{S_{T-1}\}} E_{T-1} \left\{ S_{T-1} \left(\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \right. \\
&\quad \left[\left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) + \frac{\phi \left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\alpha P_{T-2} + \beta S_{T-1} P_{T-2} - \gamma \rho O_{T-2} P_{T-2}}{\sqrt{\gamma^2 P_{T-2}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \\
&\quad + (W_{T-1} - S_{T-1}) \left(\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2} \right) \\
&\quad \left[\left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right) \right. \\
&\quad \left. + \frac{\phi \left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)}{\Phi \left(\frac{\{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\} \{\alpha + \beta(W_{T-1} - S_{T-1}) - \gamma \rho^2 O_{T-2} - \gamma \rho \eta_{T-1}\}}{\sqrt{\gamma^2 \{P_{T-2}(\alpha + 1 + \beta S_{T-1} - \gamma \rho O_{T-2} - \gamma \eta_{T-1}) + \varepsilon_{T-1}\}^2 \sigma_\eta^2 + \sigma_\varepsilon^2}} \right)} \right] \left. \right\}
\end{aligned}$$

□